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Calogero–Sutherland eigenfunctions with mixed boundary conditions and conformal field theory correlators

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Received 6 November 2006, in final form 23 January 2007

Published 21 February 2007

Online at stacks.iop.org/JPhysA/40/2509

Abstract

We construct certain eigenfunctions of the Calogero–Sutherland Hamiltonian for particles on a circle, with mixed boundary conditions. That is, the behaviour of the eigenfunction, as neighbouring particles collide, depend on the pair of colliding particles. This behaviour is generically a linear combination of two types of power laws, depending on the statistics of the particles involved. For fixed ratio of each type at each pair of neighbouring particles, there is an eigenfunction, the ground state, with lowest energy, and there is a discrete set of eigenstates and eigenvalues, the excited states and the energies above this ground state. We find the ground state and special excited states along with their energies in a certain class of mixed boundary conditions, interpreted as having pairs of neighbouring bosons and other particles being fermions. These particular eigenfunctions are characterized by the fact that they are in direct correspondence with correlation functions in boundary conformal field theory. We expect that they have applications to measures on certain configurations of curves in the statistical $O(n)$ loop model. The derivation, although completely independent of results of conformal field theory, uses ideas from the ‘Coulomb gas’ formulation.

PACS numbers: 02.50.Cw, 05.10.Gg, 11.25.Hf

1. Introduction

Recently [1], the Calogero–Sutherland quantum-mechanical Hamiltonian (see, for instance, the book [2]) was shown to be related to certain bulk-boundary correlation functions in

conformal field theory on the disc. The Calogero–Sutherland Hamiltonian for N particles at angles $\theta_1, \dots, \theta_N$ on the circle, with parameter β , is

$$H_N(\beta) = - \sum_{j=1}^N \frac{1}{2} \frac{\partial^2}{\partial \theta_j^2} + \frac{\beta(\beta-2)}{16} \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2\left(\frac{\theta_j - \theta_k}{2}\right)}. \quad (1.1)$$

The corresponding CFT has central charge c related to the parameter β through

$$\beta = \frac{8}{\kappa}, \quad c = 1 - \frac{3(4-\kappa)^2}{2\kappa}.$$

The Hamiltonian is invariant under $\beta \rightarrow 2 - \beta$. The relations above imply that we chose the range $\beta \in [1, \infty]$ for the values $\kappa \in [0, 8]$ that we will consider in this paper. This discovery initially came from an analysis of the equations believed to be associated with multiple SLE_κ processes (Schramm–Loewner evolution (SLE) processes were introduced in [3], multiple SLE generalizations were introduced in [4] and developed to a large extent in [5], although only through conjectured properties—see appendix B of a short review of what SLE is). But the connection can be established solely from CFT concepts, as was shown in [1]. The main ingredients are level-2 degenerate boundary fields, one for each particle, and a bulk primary field at the centre of the disc: the N null-vector differential operators [6] acting on the correlation functions can be recast, by taking a linear combination, into the Calogero–Sutherland Hamiltonian. Hence, the correlation functions can then be recast into eigenfunctions of the Hamiltonian. Various choices of primary field give rise to various eigenfunctions; in particular, the dimension of the bulk primary field is connected to the energy associated with the eigenfunction. But not all eigenfunctions can be reproduced in this way, since the N null-vector equations are more restrictive than that eigenvalue equation of the Hamiltonian. Two problems arise then naturally: to determine which eigenfunctions (that is, which boundary conditions, and for these boundary conditions, which states) indeed give rise to correlation functions, and to obtain explicit expressions for these eigenfunctions. These two problems are solved in great part in this paper.

Finding eigenfunctions of the Hamiltonian (1.1) requires one more piece of information: the behaviour of eigenfunctions Ψ as particles collide (boundary conditions). Fixing the boundary conditions (which we will sometimes refer to as choosing a sector) fixes the Hilbert space; we will be more precise in the text about how boundary conditions are fixed. From the CFT viewpoint, these behaviours are related to the boundary operator product expansion (OPE) (more precisely, the overlap between the bulk primary field and the boundary OPEs). Here and in the following, we choose the sector $\theta_1 > \dots > \theta_N > \theta_1 - 2\pi$, and we will consider the behaviour of eigenfunctions at the collisions $\theta_i \rightarrow \theta_{i+1}^+$ (with $\theta_{N+1} \equiv \theta_1 - 2\pi$). It will be sufficient to specify the behaviour of an eigenfunction at these boundaries in order to fix the eigenfunction³. An elementary indicial analysis of the Calogero–Sutherland system shows that the behaviour of the wavefunction as two particles collide is generically a linear combination of two types of power laws, which we will refer to as ‘bosonic’:

$$\Psi \propto (\theta_i - \theta_{i+1})^{\frac{\kappa-4}{\kappa}}, \quad (\theta_i - \theta_{i+1} \rightarrow 0^+), \quad (1.2)$$

and ‘fermionic’:

$$\Psi \propto (\theta_i - \theta_{i+1})^{\frac{4}{\kappa}} \quad (\theta_i - \theta_{i+1} \rightarrow 0^+). \quad (1.3)$$

³ The behaviour at collisions pertaining to other ordering of the angles can in principle be obtained by analytic continuation.

This nomenclature comes from the fact that for $0 < \kappa < 4$ the wavefunction vanishes as fermions (particles with fermionic boundary conditions) collide, whereas it diverges as bosons (particles with purely bosonic boundary conditions) collide. From the conformal field theory viewpoint, these correspond to the two families appearing in the fusion of level-2 null fields: that of the identity, and that of level-3 null fields. It is a simple matter to verify, for instance, that the ground state in the sector with fermionic boundary conditions at all pairs of colliding particles, which is the usual fermionic ground state, does correspond to a correlation function satisfying all null-vector equations, but the ground state with all bosonic boundary conditions generically does not.

In this paper, we solve the null-vector equations for certain bulk primary operators, of various scaling dimensions and of any spin. The results give rise to integral formulae for certain eigenfunctions of the Calogero–Sutherland Hamiltonian. These integral formulae are in close relation with those obtained by Dubédat [7], who was essentially considering the case without bulk field. The technique we use is at the basis of the Coulomb gas formalism [8, 9] of CFT for bulk correlation functions in minimal models, and works for generic central charge and for boundary operators as well. This technique was also used in [10] for correlation functions without bulk field⁴. The motivation was to evaluate the ‘auxiliary functions’ appearing in constructions of multiple SLE processes [5], which satisfy the level-2 null-vector equations with zero-dimension bulk field. In the present paper, we will derive the formulae in the simplest way possible; we do not need any of the machinery developed for the Coulomb gas formalism, for CFT or for SLE, as we work only with the differential equations.

The construction gives rise to a certain class of boundary conditions for the eigenfunctions, satisfied by the ground states and the excited states. It is important to understand that with mixed boundary conditions, one should only distinguish between classes C whose elements can be obtained from one another by simple linear combinations:

$$\Psi_1 \in C \quad \text{and} \quad \Psi_2 \in C \Rightarrow \Psi_3 = a\Psi_1 + b\Psi_2 \in C \quad \text{for} \quad a \geq 0, \quad b \geq 0.$$

If Ψ_1 and Ψ_2 have different mixed boundary conditions but correspond to the same energy, then Ψ_3 also is an eigenfunction of the Hamiltonian, with yet again different mixed boundary conditions and with the same energy. Also, if both Ψ_1 and Ψ_2 are ground states, everywhere positive, then Ψ_3 also is (this is why we need the condition that both a and b be greater than zero: the eigenfunction of a ground state should be everywhere positive). It is easy to obtain, from Ψ_1 with a given energy, Ψ_2 with the same energy: one only needs, for instance, to make cyclic permutations of the particle positions.

The classes of boundary conditions that we obtained are those with distinctive elements as follows: some pairs of colliding neighbouring particles, which do not have common members among each other, present purely bosonic behaviour (that is, the eigenfunction behaves like (1.2) times a power series in $(\theta_i - \theta_{i+1})$), pairs formed by any other neighbouring particles present purely fermionic behaviour, and the remaining pairs present both fermionic and bosonic components, in a certain fixed proportion (see figure 1). One can interpret the pairs with purely bosonic behaviour as being pairs of bosons, whereas the other particles as being fermions. Note that in one dimension this does not have any implication for the way the eigenfunction should behave when non-neighbouring particles approach each other: we work only with a fixed ordering of the $\{\theta_j\}$. While the eigenfunction may be analytically continued to other orderings, these are not physical. This is in distinction to the case in higher dimensions, when particles can be moved past each other.

The Calogero–Sutherland system was mainly studied, until now, on the Hilbert space of wavefunctions with simple uniform boundary conditions: all particle collisions giving only

⁴ We wish to note that the paper [10] was published after we had established the working of our technique.

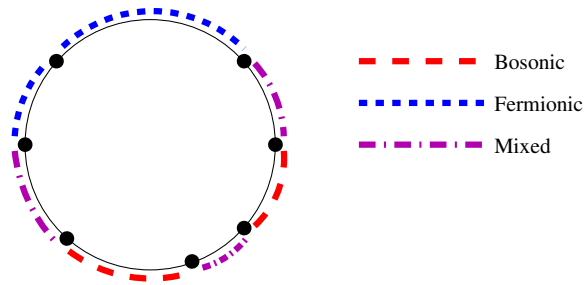


Figure 1. An example of boundary conditions satisfied by our solutions.

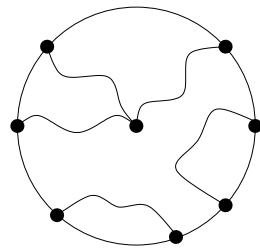


Figure 2. An example of a domain-wall configuration in the $O(n)$ model, or multiple SLE configuration, corresponding to our solution with the 3-leg operator.

fermionic exponents, or all giving only bosonic exponents. Our new solutions to the null-vector equations give special eigenfunctions of the Calogero–Sutherland system for mixed boundary conditions, physically corresponding to some particles being fermions and some being bosons. An eigenfunction of the Calogero–Sutherland Hamiltonian with nonzero (angular) momentum, which would correspond to a bulk primary field with nonzero spin, can always be obtained from one with zero momentum by a Galilean transformation. However, such eigenfunctions are not generically in agreement with all null-vector equations. Our solutions with nonzero spin are not simple Galilean transform of those with zero spin. They are yet new solutions, and correspond, in fact, to giving nonzero momentum only to the fermions.

Some of the bulk primary operators corresponding to our solutions are the ‘ N' -leg operators’ (with $N' = N - 2M$, $M \in \mathbb{N}$)—that is, their dimensions are the ‘ N' -leg’ exponents [11]. They have meaning in the context of the critical $O(n)$ loop model [12], and they are expected to be connected to certain restriction of or events in multiple-SLE measures. The corresponding solutions are expected to be related to measures for configurations of the type shown in figure 2 (although we could only give a conjecture for this relation in the case where there is single pairing). It is these solutions that lead to ground states. Other operators correspond to excited states and to states with nonzero momentum, but have as yet no known physical interpretation.

The paper is organized as follows. In section 2, we recall the results of [1]. Then, in section 3, we construct our integral representations, and derive the associated boundary conditions. We also give the Coulomb-gas interpretation of our construction. In section 4, we derive some general results about solutions to the null-vector equations in the case where $N = 3$ (we show that our solutions form a complete basis). Finally, in section 5, we discuss the interpretation of our results in the continuum $O(n)$ loop model.

2. Review of the connection between null vector equations and the Calogero–Sutherland system

2.1. Null vector equations

Consider the following family of correlation functions in a boundary conformal field theory on the unit disc:

$$G = \langle \phi(e^{i\theta_1}) \cdots \phi(e^{i\theta_N}) \Phi(0) \rangle \tag{2.1}$$

where ϕ are primary level-2 degenerate boundary fields and Φ is a primary bulk field, for $N = 1, 2, \dots$. For now we will consider only spinless bulk fields Φ , deferring the discussion of a field with spin to section 3.3. Using the parameter κ appearing naturally in SLE, we will parametrize the central charge c and the dimension h of the boundary fields by

$$c = 1 - \frac{3(4 - \kappa)^2}{2\kappa}, \quad h = \frac{6 - \kappa}{2\kappa}. \tag{2.2}$$

As was shown in [1], the null-vector equations [6, 13] associated with this correlation function imply that a certain simple modification of this correlation function is an eigenfunction of the Calogero–Sutherland Hamiltonian. One applies the infinitesimal conformal transformation $z \mapsto z + \alpha(z)$ with

$$\alpha(z) = \sum_{j=1}^N b_j \alpha_j(z), \quad \alpha_j(z) = -z \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}}. \tag{2.3}$$

Thanks to the relation

$$\overline{\alpha_j(z)} = -\bar{z}^2 \alpha_j(\bar{z}^{-1}), \tag{2.4}$$

this infinitesimal transformation preserves the region $\mathbb{D} \setminus \{z_j\}$, the disc minus the boundary points $z_j \equiv e^{i\theta_j}$. It is a pure scaling at the centre $\alpha(z) \sim z, z \rightarrow 0$, and it has poles at the positions z_j of the boundary fields, generating locally there non-trivial conformal transformations whose effect can be evaluated thanks to the null-vector property. The result is the set of differential equations

$$\sum_{j=1}^N b_j \mathcal{D}_j G = d_\Phi \left(\sum_{j=1}^N b_j \right) G \tag{2.5}$$

where d_Φ is the scaling dimension of Φ , and with

$$\mathcal{D}_j = -\frac{\kappa}{2} \left(\frac{\partial}{\partial \theta_j} \right)^2 + \frac{(6 - \kappa)(\kappa - 2)}{8\kappa} - \sum_{k \neq j} \left(\cot \left(\frac{\theta_k - \theta_j}{2} \right) \frac{\partial}{\partial \theta_k} - \frac{h}{2 \sin^2 \left(\frac{\theta_k - \theta_j}{2} \right)} \right). \tag{2.6}$$

This is derived in appendix B for completeness of the discussion. Note further that

$$\frac{(6 - \kappa)(\kappa - 2)}{8\kappa} = \frac{h}{6} + \frac{c}{12}.$$

This reproduces the operator \mathcal{D}_j obtained in [1] from slightly different arguments.

2.2. Calogero–Sutherland Hamiltonian

It will be convenient for this paper to introduce the notation:

$$f(\theta) = \cot \left(\frac{\theta}{2} \right), \quad f_{jk} = f(\theta_j - \theta_k), \quad F_j = \sum_{k \neq j} f_{jk}. \tag{2.7}$$

Using this notation, we have

$$\mathcal{D}_j = -\frac{\kappa}{2}\partial_j^2 + \frac{(6-\kappa)(\kappa-2)}{8\kappa} + \sum_{k \neq j} (f_{jk}\partial_k - hf'_{jk}) \tag{2.8}$$

with $\partial_j \equiv \partial/\partial\theta_j$, and the Calogero–Sutherland Hamiltonian (1.1) can be written as

$$H_N(\beta) = -\sum_j \left(\frac{1}{2}\partial_j^2 + \frac{\beta(\beta-2)}{16}F'_j \right). \tag{2.9}$$

In order to relate the null-vector equations to the Calogero–Sutherland system, we look at the case where $b_j = 1$ for $j = 1, \dots, N$. We will denote

$$\mathcal{D} = \sum_j \mathcal{D}_j = -\frac{\kappa}{2}\sum_j \partial_j^2 + N\frac{(6-\kappa)(\kappa-2)}{8\kappa} - \sum_j (F_j\partial_j + hF'_j). \tag{2.10}$$

Equation (2.5) implies that correlation functions G are eigenfunctions of \mathcal{D} with eigenvalue Nd_Φ . Consider the function of all θ_j 's

$$g_r = \prod_{1 \leq j < k \leq N} \left(\sin \frac{\theta_j - \theta_k}{2} \right)^{-2r} \tag{2.11}$$

$(\theta_i > \theta_{i+1}, i = 1, \dots, N-1, \theta_N > \theta_1 - 2\pi).$

From the properties $\partial_j g_r = -r g_r F_j$ and $\sum_j F_j^2 = -2\sum_j F'_j - \frac{N(N^2-1)}{3}$, it is a simple matter to check that

$$g_{-\frac{1}{\kappa}} \cdot \mathcal{D} \cdot g_{\frac{1}{\kappa}} = \kappa H_N \left(\frac{8}{\kappa} \right) - \frac{N(N^2-1)}{6\kappa} + N\frac{(6-\kappa)(\kappa-2)}{8\kappa} \tag{2.12}$$

(here and below, the dot (\cdot) means multiplication as operators on functions). Hence, any correlation function G gives rise to an eigenfunction

$$\Psi = g_{\frac{1}{\kappa}}^{-1} G \tag{2.13}$$

of the Calogero–Sutherland Hamiltonian $H_N \left(\frac{8}{\kappa} \right)$, with eigenvalue

$$E = \frac{N}{\kappa} \left[d_\Phi + \frac{(N^2-1)}{6\kappa} - \frac{(6-\kappa)(\kappa-2)}{8\kappa} \right]. \tag{2.14}$$

2.3. The fermionic and bosonic ground states

The set of null-vector equations (2.5) is more restrictive than the eigenvalue equations for the Hamiltonian (1.1). Hence, not all eigenfunctions satisfy all requirements to be associated with CFT correlation functions. Here we recall the fermionic and bosonic ground states of the Calogero–Sutherland Hamiltonian, and verify in which case they can be associated with correlation functions.

It is a simple matter to find certain eigenfunctions of the operator \mathcal{D} , which correspond to the ground state of $H_N \left(\frac{8}{\kappa} \right)$ with all fermionic (1.3) or all bosonic (1.2) boundary conditions. Indeed, we have

$$g_{-r} \cdot \mathcal{D} \cdot g_r = -\frac{\kappa}{2}\sum_j \partial_j^2 + (\kappa r - 1)\sum_j F_j\partial_j + \left(-2r + \kappa r^2 + \frac{\kappa r}{2} - h \right)\sum_j F'_j + N\frac{(6-\kappa)(\kappa-2)}{8\kappa} + \left(-r + \frac{\kappa r^2}{2} \right)\frac{N(N^2-1)}{3}$$

so that with the two values

$$r = r_f \equiv -\frac{1}{\kappa}, \quad r = r_b \equiv h = \frac{6 - \kappa}{2\kappa} \quad (2.15)$$

the factor multiplying $\sum_j F'_j$ vanishes (note that these two values are equal, $r_f = r_b$, only at $\kappa = 8$). Hence, a simple eigenfunction of this operator is 1, which gives the usual fermionic and bosonic ground-state eigenfunctions of the Calogero–Sutherland Hamiltonian (see, for instance, [2])

$$\Psi_N^f = g_{r_f - \frac{1}{\kappa}} = g_{-\frac{2}{\kappa}}, \quad \Psi_N^b = g_{r_b - \frac{1}{\kappa}} = g_{\frac{4-\kappa}{2\kappa}} \quad (2.16)$$

with the associated eigenvalues

$$E_N^f = \left(\frac{4}{\kappa}\right)^2 \frac{N(N^2 - 1)}{24}, \quad E_N^b = \left(\frac{4 - \kappa}{\kappa}\right)^2 \frac{N(N^2 - 1)}{24}. \quad (2.17)$$

They are related by the transformation $4/\kappa \rightarrow 1 - 4/\kappa$ keeping the Hamiltonian $H_N(8/\kappa)$ invariant.

The corresponding eigenfunctions of the operator \mathcal{D} are

$$G_N^f = g_{r_f}, \quad G_N^b = g_{r_b} \quad (2.18)$$

and the eigenvalues are Nd_N^f, Nd_N^b with

$$d_N^f = \frac{N^2}{2\kappa} - \frac{(\kappa - 4)^2}{8\kappa} \quad (2.19)$$

and

$$d_N^b = \frac{(6 - \kappa)(\kappa - 2)}{24\kappa} (4 - N^2). \quad (2.20)$$

Note that in general, any correlation function G with behaviour $\propto (\theta_i - \theta_{i+1})^{-2r_f}$ leads to the fermionic boundary condition (1.3) for the wavefunction, whereas any correlation function with behaviour $\propto (\theta_i - \theta_{i+1})^{-2r_b}$ leads to the bosonic boundary condition (1.2).

It is a simple matter to check that G_N^f satisfies all equations (2.5), but that G_N^b does not, unless $N = 2$ or $\kappa = 6$ or $\kappa = 8$ (in the latter case, $G_N^b = G_N^f$). Hence, G_N^f is a CFT correlation function, and d_N^f is a scaling dimension of a primary operator that couples to boundary level-two null vectors; whereas G_N^b and d_N^b generically are not. Note that d_N^f is equal to the N -leg exponent [11]. Consider the similarity transform

$$\begin{aligned} g_{-r} \cdot \mathcal{D}_j \cdot g_r &= -\frac{\kappa}{2} \partial_j^2 + \kappa r F_j \partial_j - \frac{\kappa r^2}{2} F_j^2 + \frac{\kappa r}{2} F'_j \\ &+ \sum_{k \neq j} (f_{jk} (\partial_k - r F_k) - h f'_{jk}) + \frac{(6 - \kappa)(\kappa - 2)}{8\kappa}. \end{aligned} \quad (2.21)$$

The function g_r is an eigenfunction of \mathcal{D}_j if and only if the term

$$-\frac{\kappa r^2}{2} F_j^2 + \left(\frac{\kappa r}{2} - h\right) F'_j - r \sum_{k \neq j} f_{jk} F_k + \frac{(6 - \kappa)(\kappa - 2)}{8\kappa}$$

is a constant. Some algebra (or a simple analysis of the simple and double poles) shows that it is indeed constant if and only if

$$-2\kappa r^2 - \kappa r + 2h + 4r = 0 \quad \text{and} \quad r(2\kappa r + 2) \sum_{k \neq j, k \neq l} f_{lk} = 0 \quad (2.22)$$

for all $l \neq j$. The first condition is satisfied for $r = r_f$ or $r = r_b$ only, and the second, for $N = 2$ or $r = r_f$ or $r = 0$. Hence, for $N > 2$, G_N^f is a common eigenfunction of all \mathcal{D}_j for

any κ , and G_N^b is only for $\kappa = 6$ (making $h = 0$) or $\kappa = 8$ (in which case $r_f = r_b$). In the case $\kappa = 6$, the function G_N^b is just a constant. One can also check that the eigenvalues are independent of j :

$$\mathcal{D}_j g_r = \left(\frac{\kappa r^2}{2} (N - 1)^2 - r(N - 1) + \frac{(6 - \kappa)(\kappa - 2)}{8\kappa} \right) g_r \tag{2.23}$$

in the cases above. The eigenvalues are indeed equal to $d_{\Phi_N^f}$ and $d_{\Phi_N^b}$ when we put, respectively, $r = r_f$ and $r = r_b$.

2.4. L^2 -normalizability and Hermiticity

The fermionic ground-state eigenfunction Ψ_f (2.16) is L^2 -normalizable for the full range $0 < \kappa \leq 8$, but the bosonic one, Ψ_b , is only for $8/3 < \kappa \leq 8$ (the value $\kappa = 8/3$ is the value at which the bosonic behaviour is of power $-1/2$ in wavefunctions). In general, as soon as a wavefunction has bosonic behaviour at some colliding pair of angles, it is L^2 -normalizable only in that range; in particular, this holds for all wavefunctions in sectors with mixed boundary conditions found below. For generic κ in the normalizable range, the Hamiltonian (1.1) is Hermitian. This is easy to understand from the L^2 norm. With Ψ_1 and Ψ_2 Hamiltonian eigenfunctions (possibly in a mixed sector), one evaluates $\int_\varepsilon d\theta_1 \cdots d\theta_N \Psi_1^* H \Psi_2$ where the integration region is $\theta_i > \theta_{i+1} + \varepsilon$ (and $\theta_{N+1} \equiv \theta_1 - 2\pi$) for some small positive ε . By normalizability and by the eigenfunction property, this multiple integral converges as $\varepsilon \rightarrow 0$. Checking Hermiticity involves integrating by part on all angles (to make things more obvious, one could change variables to angle differences and the total angle average), and the only possible violation of Hermiticity comes from boundary terms as angle differences are equal to ε . But for generic κ , these will be non-integer (possibly negative) powers of ε , generically not the power 0. Since the multiple integral resulting after integration by part is also convergent, all boundary terms must vanish as $\varepsilon \rightarrow 0$, which shows Hermiticity.

3. Integral representations of solutions to null-vector equations: mixed boundary conditions and excited states

In this section, we construct integral representations for solutions to the null-vector equations (2.5) employing a technique that mimics the Coulomb gas formalism of CFT. We will observe that some of these solutions correspond to excited states of the Calogero–Sutherland system above the completely fermionic ground state, that some correspond to completely new ground-state solutions with boundary conditions that are purely bosonic at certain pairs of colliding angles and purely fermionic at other pairs (as described in the introduction), and that some are excited states above these new ground states. The results of this section are very similar in form to those of Dubédat [7], and the techniques are in close relations to those used in [10].

3.1. One integration variable

Consider the function

$$w = G_N^f \prod_{1 \leq k \leq N} \left| \sin \frac{\theta_k - \zeta}{2} \right|^{-2\alpha}. \tag{3.1}$$

Denote

$$f_j = f(\theta_j - \zeta). \tag{3.2}$$

Then, we have

$$\partial_j w = \frac{F_j}{\kappa} w - \alpha f_j w. \tag{3.3}$$

Consider also the new operator

$$\mathcal{W}_j = \mathcal{D}_j + f_j \partial_\zeta - f'_j. \tag{3.4}$$

One finds that

$$\begin{aligned} (w^{-1} \cdot \mathcal{W}_j \cdot w)1 &= \frac{(6 - \kappa)(\kappa - 2)}{8\kappa} + \frac{N^2 - 1}{2\kappa} + \left(\frac{\kappa\alpha}{2} - 1\right) f'_j \\ &\quad - \alpha \left(\frac{\kappa\alpha}{2} - 1\right) f_j^2 - \alpha \sum_{k \neq j} (f_{jk}(f_k - f_j) - f_k f_j). \end{aligned} \tag{3.5}$$

In order to cancel the double pole at $\theta_j = \beta$ coming from f'_j and f_j^2 , we need

$$\alpha \left(\frac{\kappa\alpha}{2} - 1\right) = -\frac{1}{2} \left(\frac{\kappa\alpha}{2} - 1\right) \tag{3.6}$$

so that

$$\alpha = -\frac{1}{2} \quad \text{or} \quad \alpha = \frac{2}{\kappa}. \tag{3.7}$$

It is a simple matter to verify that the sum $\sum_{k \neq j} (f_{jk}(f_k - f_j) - f_k f_j)$ does not have poles at $\theta_j = \zeta, \theta_j = \theta_k (k \neq j)$ and $\theta_k = \zeta (k \neq j)$, so that it is a constant. Evaluating this constant by taking $\theta_j \rightarrow -i\infty$, where $f_{jk} = f_j = i$, we find

$$\mathcal{W}_j w = \left[\frac{N^2}{2\kappa} - \frac{(\kappa - 4)^2}{8\kappa} + \alpha \left(\frac{\kappa\alpha}{2} - N\right) \right] w \equiv d_N^{(\alpha)} w. \tag{3.8}$$

In the first case of (3.7), the eigenvalue is given by

$$d_N^{(-\frac{1}{2})} = \frac{(N + \kappa/2)^2}{2\kappa} - \frac{(\kappa - 4)^2}{8\kappa} \tag{3.9}$$

whereas in the second case, it is

$$d_N^{(\frac{2}{\kappa})} = \frac{(N - 2)^2}{2\kappa} - \frac{(\kappa - 4)^2}{8\kappa} = d_{N-2}^f. \tag{3.10}$$

We now consider the analytic continuation of w as function of β . For definiteness, we choose the analytic continuation from the region $\theta_N > \beta > \theta_1 - 2\pi$, where it is real and positive, and we still denote this analytic continuation by w . Note that

$$\mathcal{W}_j = \mathcal{D}_j + \partial_\zeta \cdot f_j. \tag{3.11}$$

Hence, the function

$$G_C = A \oint_C d\zeta w \tag{3.12}$$

satisfies

$$\mathcal{D}_j G_C = d_N^{(\alpha)} G_C \tag{3.13}$$

for any closed contour \mathcal{C} on the multi-sheeted Riemann surface on which w lives as a function of ζ ; the function G_C will be nonzero only for contours that are topologically non-trivial. The normalization constant A will be chosen for convenience: if possible, it will be such that the result is real and positive in the chosen sector $\theta_1 > \dots > \theta_N > \theta_1 - 2\pi$. This is necessary for identifying the result as a ground state of the Calogero–Sutherland system (that is, without zeros), as well as for its interpretation as a measure on stochastic processes (but obviously not

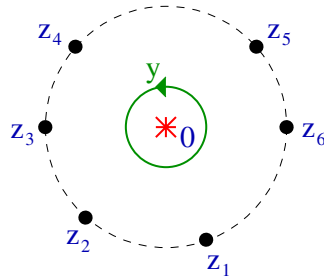


Figure 3. Contour C_{origin} (with $N = 6$).

necessary for the interpretation as correlation functions, or as linear combinations of measures with complex coefficients).

In fact, the analytic structure of the integration measure $d\zeta w$ is easier to see when it is expressed in terms of the variables $z_j = e^{i\theta_j}$ and $y = e^{i\zeta}$. In terms of these variables, the function w is

$$w = G_N^f (2i)^{2\alpha N} y^{\alpha N} \prod_{1 \leq k \leq N} [z_k^\alpha (z_k - y)^{-2\alpha}]. \tag{3.14}$$

The analytic structure of the function $d\beta/dyw = -iw/y$ is as follows.

- *Case $\alpha = -\frac{1}{2}$.* There are two singular points: one at $y = 0$, of the type $y^{-1-N/2}[[y]]$, and the other at $y = \infty$, of the type $y^{-1+N/2}[[y^{-1}]]$.
- *Case $\alpha = \frac{2}{\kappa}$.* There are singular points at $y = z_j$ of the type $(y - z_j)^{-4/\kappa}[[y - z_j]]$, at $y = 0$ of the type $y^{-1+2N/\kappa}[[y]]$ and at $y = \infty$ of the type $y^{-1-2N/\kappa}[[y^{-1}]]$.

3.1.1. *Case $\alpha = -\frac{1}{2}$: excited state above the fermionic ground state.* If N is even, there is only one class of topologically non-trivial contours, those circling the origin. Circling once counterclockwise (contour C_{origin} , see figure 3), the result is (with appropriate normalization)

$$\begin{aligned} G_{C_{\text{origin}}} &= G_N^f \sum_{\substack{u \equiv \{u_1, \dots, u_{N/2}\}, v \equiv \{v_1, \dots, v_{N/2}\} \\ |u \cup v| = \{1, \dots, N\}}} \cos \sum_{j=1}^{N/2} \frac{\theta_{u_j} - \theta_{v_j}}{2} \\ &= \frac{1}{(N/2)!} G_N^f \sum_{\substack{\{\{u_1, v_1\}, \dots, \{u_{N/2}, v_{N/2}\}\} \\ |\cup_{j=1}^{N/2} \{u_j, v_j\}| = \{1, \dots, N\}}} \prod_{j=1}^{N/2} \cos \frac{\theta_{u_j} - \theta_{v_j}}{2}. \end{aligned} \tag{3.15}$$

Observe that although the result is clearly real, it is impossible to make it positive everywhere (for any one ordering of the angles).

If N is odd, there is only one class again, a representative being the 8-shaped contour circling the origin counterclockwise and the point ∞ clockwise (or circling twice the origin counterclockwise). However, because of the structure of the singularities at the origin and at infinity, this gives zero: there are no contours giving nonzero eigenfunctions with eigenvalue $d_N^{\frac{1}{2}}$ for N odd.

The dimension $d_N^{(-\frac{1}{2})}$ associated with the correlation function G (3.15) corresponds to the energy of a certain excited state of the Calogero–Sutherland Hamiltonian above the N -particle completely fermionic ground state Ψ_N^f (2.16) for N even. In general, these excited states are characterized by a set of non-negative integers p_j , $j = 1, \dots, N - 1$, and have eigenvalues

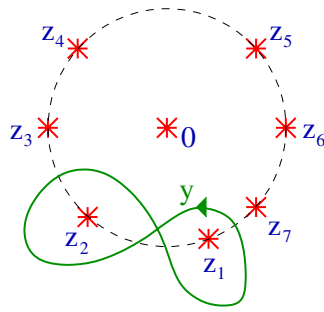


Figure 4. Contour $C^{(1)}$ (with $N = 7$).

$E_N^{f;p_1, \dots, p_{N-1}} = \frac{1}{2} \sum_{j=1}^N k_j^2$ with $\sum_{j=1}^N k_j = 0, k_{j+1} - k_j = 4/k + p_j$ [2]. The corresponding field dimension, related to the eigenvalues $E_N^{f;p_1, \dots, p_{N-1}}$ through (2.14), will be denoted as $d_N^{f;p_1, \dots, p_{N-1}}$. A configuration of p_1, \dots, p_{N-1} that reproduces the dimension $d_N^{(-\frac{1}{2})}$ satisfies

$$\sum_{l=1}^{N-1} \sum_{l'=1}^{N-1} (N \min(l, l') - ll') p_l p_{l'} = \frac{N^2}{4}$$

$$\sum_{l=1}^{N-1} l(N-l) p_l = \frac{N^2}{4}.$$

It is a simple matter to observe that there are no solutions for odd N , and that for all N even,

$$d_N^{(-\frac{1}{2})} = d_N^{f;p_1, \dots, p_{N-1}}$$

$$\text{with } \begin{cases} p_1 = \dots = p_{N/2-1} = p_{N/2+1} = \dots = p_{N-1} = 0, \\ p_{N/2} = 1 \end{cases} \quad (3.16)$$

(N even).

Also, one can check that there are no other configurations of p_1, \dots, p_{N-1} that reproduce $d_N^{(-\frac{1}{2})}$ for all even $N \leq 10$ (that is, these states are non-degenerate).

It is easy to check explicitly that our solution (3.15) for N even, along with the transformation (2.13), reproduces the well-known eigenfunctions for these excited states. Indeed, the function multiplying G_N^f is proportional to the sum over all z_i -permutations of the product $\prod_i z_i^{\lambda_i}$ where half of the λ_i 's are $+1/2$, and half are $-1/2$. This corresponds to a single gap in the ‘Fermi sea’ of particles, making it two filled bands with the same number of particles separated by the minimum energy, as described by the configuration $p_{N/2} = 1$ and $p_j = 0, j \neq N/2$, along with the constraint of zero total momentum (see, for instance, [2]).

Hence we have found that certain fermionic excited states of the Calogero–Sutherland Hamiltonian are in fact also solutions to all null-vector equations (2.5).

3.1.2. Case $\alpha = \frac{2}{\kappa}$: ground state with mixed boundary conditions. In this case, many classes of non-trivial contours exist. It turns out that a basis can be obtained by taking 8-shaped contours that surround the point z_i once counterclockwise and the point z_{i+1} once clockwise, for $i = 1, \dots, N$ (with $z_{N+1} \equiv z_1$). We will denote contours of this type by $C^{(i)}$ where i stands for the index of the first member z_i (in the clockwise ordering around the unit circle) of a pair of adjacent angles (see figure 4). The result of the integration with any other contour can be written as a linear combination of the integration with the contours $C^{(i)}$. Note that 8-shaped

contours are closed since the singularities at the points z_1, \dots, z_N are all of the same type. The solutions that we consider are then

$$G_{C^{(i)}} = A(2i)^{\frac{4N}{\kappa}} G_N^f \int_{C^{(i)}} dy y^{\frac{2N}{\kappa}-1} \prod_{1 \leq j \leq N} \left(z_j^{\frac{2}{\kappa}} (z_j - y)^{-\frac{4}{\kappa}} \right). \tag{3.17}$$

The two parts of the integration contour that lie between z_i and z_{i+1} can be collapsed to a segment of line (on different Riemann sheets), and if $\kappa > 4$, the contributions around the points z_i and z_{i+1} can be set to zero by collapsing them upon the points z_i and z_{i+1} , respectively. One is then left with

$$G_{C^{(i)}} = A(2i)^{\frac{4N}{\kappa}} (1 - \omega) G_N^f \int_{z_i}^{z_{i+1}} dy y^{\frac{2N}{\kappa}-1} \prod_{1 \leq j \leq N} \left(z_j^{\frac{2}{\kappa}} (z_j - y)^{-\frac{4}{\kappa}} \right) \quad (\kappa > 4) \tag{3.18}$$

where

$$\omega = e^{\frac{8i\pi}{\kappa}}.$$

For the rest of this sub-section, we will restrict ourselves to the case $\kappa > 4$.

This solution corresponds to the ground state of the Calogero–Sutherland system with certain mixed boundary condition. That it is a ground state (that it has no zeros in the sector that we consider) is seen by writing expression (3.18), with an appropriate normalization constant, in a form that is obviously real and positive ($A = 1/2e^{i\varphi}$ for some real φ that depends on i):

$$(\kappa > 4) \quad G_{C^{(i)}} = \sin \frac{4\pi}{\kappa} G_N^f \int_{\theta_i}^{\theta_{i+1}} d\zeta \prod_{1 \leq k \leq N} \left| \sin \frac{\theta_k - \zeta}{2} \right|^{-\frac{4}{\kappa}}. \tag{3.19}$$

Below we show that it has purely bosonic behaviour as $\theta_i \rightarrow \theta_{i+1}$, mixed behaviour as $\theta_{i-1} \rightarrow \theta_i$ and $\theta_{i+1} \rightarrow \theta_{i+2}$, and purely fermionic elsewhere. That is, the solution takes the following forms when expanded around different pairs of colliding angles:

$$G_{C^{(i)}} = \begin{cases} (\theta_i - \theta_{i+1})^{-2r_b} [[\theta_i - \theta_{i+1}]] \\ (\theta_{i+1} - \theta_{i+2})^{-2r_b} [[\theta_i - \theta_{i+1}]] + (\theta_{i+1} - \theta_{i+2})^{-2r_f} [[\theta_{i+1} - \theta_{i+2}]] \\ (\theta_{i-1} - \theta_i)^{-2r_b} [[\theta_{i-1} - \theta_i]] + (\theta_{i-1} - \theta_i)^{-2r_f} [[\theta_{i-1} - \theta_i]] \\ (\theta_j - \theta_{j+1})^{-2r_f} [[\theta_j - \theta_{j+1}]] \end{cases} \tag{3.20}$$

where $j \neq i, i + 1, i - 1 \pmod N$.

Let us analyse the boundary conditions from expression (3.18). Take for simplicity $i = 1$ (other cases are obtained by a cyclic permutation of the variables). The singularity as $z_1 \rightarrow z_2$ can be obtained by setting the variables y and z_1 to z_2 everywhere except in the factors $(z_1 - y)^{2/\kappa}$ and $(z_2 - y)^{2/\kappa}$ and by calculating

$$\int_{z_1}^{z_2} dy (y - z_1)^{-\frac{4}{\kappa}} (z_2 - y)^{-\frac{4}{\kappa}} = \frac{\Gamma(1 - \frac{4}{\kappa})^2}{\Gamma(2 - \frac{8}{\kappa})} (z_2 - z_1)^{1 - \frac{8}{\kappa}} \tag{3.21}$$

which, multiplied by $(z_2 - z_1)^{2/\kappa}$ coming from the factor G_N^f , gives

$$\propto (z_2 - z_1)^{1 - \frac{6}{\kappa}} = (z_2 - z_1)^{-2r_b}. \tag{3.22}$$

This is the bosonic behaviour. It is in fact a purely bosonic behaviour (corrections are positive integer powers of $z_2 - z_1$), and the exact leading part of G_{C_i} is given by (taking the normalization as in (3.19))

$$G_{C^{(1)}} = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma(1 - \frac{4}{\kappa})^2}{\Gamma(2 - \frac{8}{\kappa})} G_{N-2}^f(\theta_3, \dots, \theta_N) \left(\sin \frac{\theta_1 - \theta_2}{2} \right)^{-2r_b} (1 + O(\theta_1 - \theta_2)). \tag{3.23}$$

Here we wrote everything back in terms of the angular variables, and we wrote explicitly the dependence of G_{N-2}^f on these variables for clarity. The behaviours as $z_2 \rightarrow z_3$ and $z_N \rightarrow z_1$ are generically modified: they have a bosonic part and a fermionic part. For the leading bosonic behaviour as $z_2 \rightarrow z_3$, for instance, one just replaces the variables z_2 and y by z_3 , except in the factors $(z_2 - y)^{-4/\kappa}$, $(z_3 - y)^{-4/\kappa}$ and in the integration limit z_2 . Taking the integration limit $z_1 \rightarrow \infty$ gives the leading bosonic behaviour (again with the normalization as in (3.19))

$$G_{\mathcal{C}^{(i)}} = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma(1 - \frac{4}{\kappa}) \Gamma(-1 + \frac{8}{\kappa})}{\Gamma(\frac{4}{\kappa})} G_{N-2}^f(\theta_1, \theta_4, \dots, \theta_N) \left(\sin \frac{\theta_2 - \theta_3}{2} \right)^{-2r_b} \times (1 + O(\theta_2 - \theta_3)) + O((\theta_2 - \theta_3)^{-2r_f}). \tag{3.24}$$

The fermionic part (which is subleading) has a more complicated expression that we will not write here, and is generically nonzero. We expect that, from the viewpoint of the Calogero–Sutherland Hamiltonian, the behaviours (3.23) and (3.24) (with the explicit constant for the fermionic behaviour in (3.24)) fixes the Hilbert space. Note that this is a different Hilbert space than the usual fermionic or bosonic ones, hence the new ground state (3.19) does not violate the unicity of the known fermionic or bosonic ground states of the Calogero–Sutherland Hamiltonian.

It is worth noting, however, that in the case $N = 3$ and $\kappa = 6$, the functions $G_{\mathcal{C}^{(i)}}$ all degenerate to constants (that is, in this case all behaviours are purely bosonic); this is just the solution G_3^b .

3.2. Many integration variables

It is a simple matter to extend the method to integral formulae with many integration variables. Consider now

$$w = G_N^f \prod_{1 \leq j < k \leq M} \left| \sin \frac{\zeta_j - \zeta_k}{2} \right|^{-2\beta_{jk}} \prod_{1 \leq j \leq N, 1 \leq k \leq M} \left| \sin \frac{\theta_j - \zeta_k}{2} \right|^{-2\alpha_k}. \tag{3.25}$$

A calculation similar to that of the previous sub-section shows that this is an eigenfunction of the operators

$$\mathcal{W}_j = \mathcal{D}_j + \sum_{k=1}^M (f_j^k \partial_{\zeta_k} - (f_j^k)') \tag{3.26}$$

where $f_j^k = f(\theta_j - \zeta_k)$, if and only if

$$\beta_{jk} = -\kappa \alpha_j \alpha_k \quad \text{and} \quad \left[\alpha_j = -\frac{1}{2} \text{ or } \alpha_j = \frac{2}{\kappa} \right] \tag{3.27}$$

for all $j, k = 1, \dots, M$ (and in general, we can have $\alpha_j \neq \alpha_k$ for $j \neq k$). Let us denote by Q the number of parameters α_j that are set to $-\frac{1}{2}$, and by R the number of parameters α_j that are set to $\frac{2}{\kappa}$ (that is, $Q + R = M$). Then, the eigenvalue associated with w is

$$d_N^{(Q,R)} = \frac{1}{2\kappa} \left(N - 2R + \frac{\kappa}{2} Q \right)^2 - \frac{(\kappa - 4)^2}{8\kappa} \tag{3.28}$$

(concerning the relation with our previous notation, we have $d_N^{(-\frac{1}{2})} = d_N^{(1,0)}$, $d_N^{(\frac{2}{\kappa})} = d_N^{(0,1)}$). Again, we can construct eigenfunctions of all operators \mathcal{D}_j with the eigenvalue above by

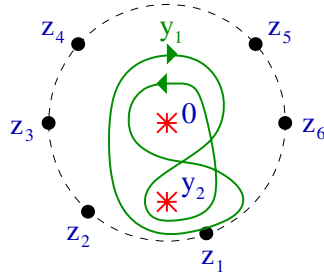


Figure 5. The y_1 part of the set of contours $\mathcal{C}_{\text{origin}}^2$ (with $N = 6$).

considering the analytic continuation of the function w (on a branch of our choice) and by constructing

$$G_{\mathcal{C}_1, \dots, \mathcal{C}_M} = A \int_{\mathcal{C}_1} d\zeta_1 \cdots \int_{\mathcal{C}_M} d\zeta_M w \tag{3.29}$$

where the contours $\mathcal{C}_1, \dots, \mathcal{C}_M$ must be topologically non-trivial. Introducing the variables $y_k = e^{i\zeta_k}$ and $z_j = e^{i\theta_j}$ will simplify the discussion of the contours. We then have

$$w = G_N^f(2i)^\nu \prod_{1 \leq k \leq M} y_k^{\alpha_k(N-2R+\frac{\kappa}{2}Q+\kappa\alpha_k)} \times \prod_{1 \leq j \leq N, 1 \leq k \leq M} [z_j^{\alpha_k} (z_j - y_k)^{-2\alpha_k}] \prod_{1 \leq j < k \leq M} (y_j - y_k)^{2\kappa\alpha_j\alpha_k}$$

and

$$G_{\mathcal{C}_1, \dots, \mathcal{C}_M} = A \int_{\mathcal{C}_1} \frac{dy_1}{y_1} \cdots \int_{\mathcal{C}_M} \frac{dy_M}{y_M} w \tag{3.30}$$

with

$$\nu = -\frac{\kappa^2 Q(Q-1) + 4\kappa Q(N-2R) + 16R(R-N-1)}{4\kappa}.$$

3.2.1. Case $R = 0$: excited states above the fermionic ground state. Taking $R = 0$, there are no singularities at the points z_1, \dots, z_N , but there are at $y_j = y_k, j \neq k$ and at $y_k = 0$. The contours \mathcal{C}_k 's in y_k -planes can be made non-trivial by ‘surrounding themselves’ and surrounding the origin, in a way that generalizes the case $Q = 1$ discussed in the previous sub-section. For instance, one may first integrate over y_1 surrounding the origin and the point y_2 in the ‘double 8’ contour shown in figure 5 (note that an 8-shaped contour is not closed in this case, since the origin and y_2 are algebraic singularities with different powers). Then, one may integrate the variable y_2 surrounding the origin and y_3 , etc, until only the variable y_Q is left. The remaining integral is of the form $\int dy_Q/y_Q y_Q^{-QN/2} [[y_Q]]$ (by power counting), which is nonzero only if QN is even; the contour can then be taken surrounding the origin. This set of contours $\mathcal{C}_1, \dots, \mathcal{C}_Q$ will be denoted as $\mathcal{C}_{\text{origin}}^Q$. One can check that explicit calculations give zero for any odd N , hence the condition for having nonzero results is that N be even.

Following the discussion in the paragraph above (3.16), a configuration of p_1, \dots, p_{N-1} that reproduces the dimension $d_N^{(Q,0)}$ satisfies

$$\sum_{l=1}^{N-1} \sum_{l'=1}^{N-1} (N \min(l, l') - l'l') p_l p_{l'} = \frac{N^2 Q^2}{4}$$

$$\sum_{l=1}^{N-1} l(N-l)p_l = \frac{N^2 Q}{4}.$$

From this, it is simple to check that for all even N ,

$$d_N^{(Q,0)} = d_N^{f;p_1,\dots,p_{N-1}}$$

with
$$\begin{cases} p_1 = \dots = p_{N/2-1} = p_{N/2+1} = \dots = p_{N-1} = 0, \\ p_{N/2} = Q \end{cases} \quad (3.31)$$

(N even).

For $N = 2$ these states are not degenerate, and for $N = 3$ there are no configurations of p_1, p_2 that would give $d_3^{(Q,0)}$. However, the situation is more complicated for higher N . For $N = 4$, the states with $Q = 3, 6, 9, 12, \dots$ are degenerate, hence there are other configurations of p_1, \dots, p_{N-1} giving $d_4^{(Q,0)}$; for $N = 5$ there are configurations for $Q = 8, 16, \dots$. However, we expect that our solution (3.30) with the contour C_{origin}^Q gives the states described by (3.31). A direct proof of this, for all N and Q , is beyond the scope of this paper. The result of the integrals gives, for instance in the case $Q = 2$ (up to a phase),

$$G_{C_{\text{origin}}^2} = A G_N^f (2i)^v \sum_{p=0}^N \frac{\Gamma(-\frac{\kappa}{4} - \frac{N}{2} + p) \Gamma(\frac{\kappa}{2} + 1)}{\Gamma(\frac{\kappa}{4} - \frac{N}{2} + 1 + p)} \sum_{\substack{u \subset \{1, \dots, N\}, v \subset \{1, \dots, n\} \\ |u|=|v|=p}} \prod_{j=1}^N \frac{z_{v_j}}{z_{u_j}}. \quad (3.32)$$

One can see that the formula above is exactly zero for N odd, as claimed.

We conjecture that with all integers $Q > 0$, the fermionic excited states of the Calogero–Sutherland Hamiltonian characterized by the sets $\{p_i\}$ as in (3.31) are all excited states above the fermionic ground state that satisfy simultaneously all null-vector equations.

3.2.2. Case $Q = 0$: ground state with mixed boundary conditions. Taking $Q = 0$, there are singularities at the points z_1, \dots, z_N . In a fashion similar to what we did in the previous sub-section, we can form pairs by integrating each of the variables y_j around two points z_{i_j} and $z_{i_{j+1}}$ in the 8-shaped contour $C^{(i_j)}$, in such a way that the contours do not cross each other. Again, with $\kappa > 4$, the contours can be collapsed to lines joining adjacent angles. This modifies the boundary conditions as in (3.20), for all indices i_1, \dots, i_R involved: the condition is purely bosonic when two angles of a same pair collide, it is mixed when the angles are part of different pairs or one of them only is part of a pair, and purely fermionic when both angles are not part of any pairs. There are other contours giving linearly independent functions (and other complicated boundary conditions), but we will not analyse them here (we do not expect them, generically, to give rise to real positive solutions). We believe that the contours described here may give rise to functions G_{C_1, \dots, C_R} with a simple stochastic interpretation, that we develop in the next section.

The expressions can be made obviously real and positive for $\kappa > 4$, hence these are ground states (here again, choosing $A = (1/2)^R e^{i\varphi}$ for some real φ depending on the indices i_1, \dots, i_R):

$$G_{C^{(i_1)}, \dots, C^{(i_R)}} = \left(\sin \frac{4\pi}{\kappa}\right)^R G_N^f \int_{\theta_{i_1}}^{\theta_{i_1+1}} d\zeta_1 \cdots \int_{\theta_{i_R}}^{\theta_{i_R+1}} d\zeta_R$$

$$\times \prod_{1 \leq j < k \leq M} \left| \sin \frac{\zeta_j - \zeta_k}{2} \right|^{\frac{8}{\kappa}} \prod_{1 \leq j \leq N, 1 \leq k \leq M} \left| \sin \frac{\theta_j - \zeta_k}{2} \right|^{-\frac{4}{\kappa}} \quad (\kappa > 4) \quad (3.33)$$

for $i_a \in \{1, \dots, N\}$, $i_{a+1} - i_a \geq 2$ and if $i_1 = 1$ then $i_R < N$. The expansions at colliding angles are of the same form as those of the previous sub-section

$$G_{C^{(1)}, C^{(i_2)}, \dots, C^{(i_R)}} = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma\left(1 - \frac{4}{\kappa}\right)^2}{\Gamma\left(2 - \frac{8}{\kappa}\right)} G_{C^{(i_2)}, \dots, C^{(i_R)}}(\theta_3, \dots, \theta_N) \times \left(\sin \frac{\theta_1 - \theta_2}{2}\right)^{-2r_b} (1 + O(\theta_1 - \theta_2)) \tag{3.34}$$

and

$$G_{C^{(1)}, C^{(i_2)}, \dots, C^{(i_R)}} = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma\left(1 - \frac{4}{\kappa}\right) \Gamma\left(-1 + \frac{8}{\kappa}\right)}{\Gamma\left(\frac{4}{\kappa}\right)} \times \left\{ \begin{array}{l} G_{C^{(i_2)}, \dots, C^{(i_R)}}(\theta_1, \theta_4, \dots, \theta_N) \quad (i_2 \neq 3) \\ G_{C^{(1)}, C^{(i_3)}, \dots, C^{(i_R)}}(\theta_1, \theta_4, \dots, \theta_N) \quad (i_2 = 3) \end{array} \right\} \times \left(\sin \frac{\theta_2 - \theta_3}{2}\right)^{-2r_b} (1 + O(\theta_2 - \theta_3)) + O((\theta_2 - \theta_3)^{-2r_f}). \tag{3.35}$$

Here we wrote explicitly the dependence on the angles where necessary for clarity. Again, we expect that, from the viewpoint of the Calogero–Sutherland Hamiltonian, the behaviours (3.34) and (3.35) (with the explicit constant for the fermionic behaviour in (3.35)) fixes the Hilbert space.

3.2.3. *Case $Q \neq 0, R \neq 0$: excited states with mixed boundary conditions.* Finally, it is a simple matter to combine the two types of contours mentioned above in the general case. Functions of the type

$$G_{C_{\text{origin}}^Q, C^{(i_1)}, \dots, C^{(i_R)}}(\theta_1, \dots, \theta_N) \tag{3.36}$$

(again, for $i_a \in \{1, \dots, N\}$, $i_{a+1} - i_a \geq 2$ and if $i_1 = 1$ then $i_R < N$) satisfy all null-vector equations (2.5), as well as the boundary conditions (3.20) for all indices i_1, \dots, i_R involved. They describe certain excited states above the mixed ground state corresponding to (3.33), described in the previous paragraph, generalizing the excited states (3.31). Obviously, these are not the only contours that can be taken, but we believe that these contours give rise to functions $G_{C_{\text{origin}}^Q, C^{(i_1)}, \dots, C^{(i_R)}}$ that describe all possible (zero-momentum) excited states above the mixed ground states (3.33) that can be obtained from our general integrable formula (3.30) (but note that the construction that we explain in the next sub-section gives yet other zero-momentum excited states). Also, they are not all possible excited states above this mixed ground state, but we conjecture that they are all (zero-momentum) excited states that satisfy simultaneously all null-vector equations.

3.3. *Solutions with nonzero total momentum or spin*

A solution to the Calogero–Sutherland eigenvalue equation with nonzero total momentum is simply obtained by multiplying a zero-momentum solution by the exponential $e^{is \sum_j \theta_j}$. The energy then gets added by the term $\frac{Ns^2}{2}$, and the total momentum is just Ns . Generally, multiplying by this exponential factor a solution of energy E and momentum P gives a new solution of energy $E + \frac{Ns^2}{2} + sP$ and momentum $P + Ns$. Hence this corresponds to adding a momentum s to that of each particle (making the quantum-mechanical average of the momentum of each particle exactly what it was before plus the value s), and the multiplication by this phase factor is just the Galilean transformation of the initial eigenfunction. Note that in order for the eigenfunction to be still defined on the circle, the total momentum must be

an integer, hence we must have $Ns \in \mathbb{Z}$ (otherwise, one may in fact interpret the particles as anyons confined on a circle).

The total momentum operator $-i \sum_j \partial_j$ is, on correlation functions, the operator for the spin of the bulk field. This would suggest that we would be able to construct in this way correlation functions corresponding to bulk fields with nonzero spin. However, such Galilean-transformed eigenfunctions do not generically give rise to solutions to all null-vector equations, since the operators \mathcal{D}_j transform non-trivially under Galilean transformation. There is a way, though, to obtain solutions to the null-vector equations that carry a nonzero spin, corresponding to nonzero total momentum for the eigenfunctions. Consider the transformation property

$$e^{-is \sum_j \theta_j} \mathcal{D}_j e^{is \sum_j \theta_j} = \mathcal{D}_j - i\kappa \partial_j + is F_j + \frac{\kappa s^2}{2}. \tag{3.37}$$

With

$$\Gamma = \sum_j \theta_j + \gamma \zeta$$

for some number γ , we then have

$$e^{-is\Gamma} \cdot \mathcal{W}_j \cdot e^{is\Gamma} = \mathcal{W}_j - i\kappa \partial_j + is F_j + \frac{\kappa s^2}{2} + is \gamma f_j \tag{3.38}$$

where \mathcal{W}_j is defined in (3.4). Note that

$$(\kappa \partial_j - F_j - \gamma f_j)w = -(\kappa \alpha + \gamma) f_j w$$

where w is defined in (3.1). Hence, choosing

$$\gamma = -\alpha \kappa = \begin{cases} \frac{\kappa}{2} & (\alpha = -\frac{1}{2}) \\ -2 & (\alpha = \frac{2}{\kappa}) \end{cases} \tag{3.39}$$

gives

$$\mathcal{W}_j(e^{is\Gamma} w) = \left(\lambda + \frac{\kappa s^2}{2} \right) e^{is\Gamma} w \tag{3.40}$$

where λ is the eigenvalue (3.9) or (3.10).

Hence, a solution with nonzero spin is obtained by replacing w by $e^{is\Gamma} w$ in the integral expression (3.12). The field dimension gets added by the term $\frac{\kappa s^2}{2}$, and the energy, by the term $\frac{Ns^2}{2}$. The same structure works for many integration variables: one needs to replace w by $e^{is(\sum_j \theta_j - \kappa \sum_k \alpha_k \zeta_k)} w$ in (3.29), and the field dimension and energy get added by the same terms. Let us denote generically the resulting correlation function by $G_s^{(Q,R)}$, employing the notation Q and R as in the paragraph above (3.28). Then, the spin of the bulk field, equivalently the total momentum of the eigenfunction, is given by $(N + \frac{\kappa Q}{2} - 2R) s$:

$$-i \sum_{j=1}^N \partial_j G_s^{(Q,R)} = \left(N + \frac{\kappa Q}{2} - 2R \right) s G_s^{(Q,R)}. \tag{3.41}$$

Note that it is *not* just Ns .

When $Q = 0$, it is like giving momentum s to each of the $N - 2R$ fermions (the particles that are not paired by bosonic boundary conditions), and giving no average momentum to the bosons (the particles that are paired). We believe that it is indeed what happens if the contours are chosen as in the discussion in the previous sub-sections. These contours are still valid, since the singular points surrounded by the integration contours of the R variables with $\alpha = \frac{2}{\kappa}$ are not affected by the extra factors coming from $e^{is\Gamma}$. However, in the case $Q = 0$,

there are still other non-trivial contours for discrete ranges of s (for instance, with $R = 1$, a necessary condition is: whether $2N/\kappa - 2s \in \mathbb{Z}$ and $s \leq N/\kappa$, or $-2N/\kappa - 2s \in \mathbb{Z}$ and $s \geq -N/\kappa$): the contours $\mathcal{C}_{\text{origin}}$ that circle the origin, or similar contours that circle the point at infinity. These contours do not affect the boundary conditions, and if $R - R'$ variables are taken with such contours, we have new nonzero-momentum solutions with $2R' < 2R$ bosons and $N - 2R' > N - 2R$ fermions.

When $Q \neq 0$, the situation becomes even more complicated. The Q integration variables associated with $\alpha = -\frac{1}{2}$ now live on Riemann surfaces with more complicated singularity structures at the origin and at infinity. The conditions for having non-trivial integration contours are generically affected. For instance, in the case $Q = 1$ and $R = 0$, a necessary condition is $\kappa s/2 \equiv q \in \mathbb{Z} + N/2$ and $-N/2 \leq q \leq N/2$. The general case $Q \neq 0, R \neq 0$ should comprise a myriad of contours, including those we described in the previous sub-sections as well as those we described here, along with conditions on the spins.

Let us note here that in order for the eigenfunction to be well defined on the circle we need

$$\left(N + \frac{\kappa Q}{2} - 2R\right)s \in \mathbb{Z}. \tag{3.42}$$

It is important to realize that this condition may not be in agreement with the conditions on s that arose above for having certain non-trivial contours. However, it is always in agreement with the contours of the previous sub-sections in the cases where $Q = 0$. That is, the eigenfunction with only the $N - 2R$ fermions being given an average momentum s is a valid one.

Condition (3.42) is not really necessary from the viewpoint of correlation functions: it is conceivable that a correlation function acquires a phase when the positions of the null fields are all brought around the circle (this would correspond to the bulk field being ‘semi-local’ with respect to the boundary null-fields).

From the viewpoint of eigenfunctions of the Calogero–Sutherland Hamiltonian, we can now apply a Galilean transformation to bring the momentum back to zero, and we obtain *different* zero-momentum solutions with a *different* energy from those corresponding to $G_0^{(Q,R)}$. These should not correspond to ground states, because they have no reason to be real and positive. The energy is given by (we denote by $E_N^{(Q,R)}$ the energy corresponding to the dimension $d_N^{(Q,R)}$ defined in (3.28))

$$E_N^{(Q,R)} + \frac{Ns^2}{2} \left[1 - \left(1 + \frac{\kappa Q}{2N} - \frac{2R}{N}\right)^2\right]. \tag{3.43}$$

We do not fully understand yet the meaning of these new zero-momentum solutions. Certainly, for $R = 0$ these are yet other excited states above the fermionic ground states; hence, we have here integral representations for these other excited states (and these should agree, of course, with the known eigenfunctions). We have not fully identified them, because we have not fully determined the conditions on s for all $Q > 0$.

With $R \neq 0$ and $Q \neq 0$, we obtain new excited states above the ground states with mixed boundary conditions by taking the contours of the R integration variables associated with $\alpha = \frac{2}{\kappa}$ as in the discussion in the previous sub-sections, and with $s \neq 0$ restricted by the conditions coming from the Q integration variables associated with $\alpha = -\frac{1}{2}$. But, we can also take some of the R variables to have contours surrounding the origin or infinity, as described above, as long as the resulting conditions on the spin are in agreement with those coming from the Q variables associated with $\alpha = -\frac{1}{2}$ (and if $Q = 0$, there is no agreement conditions). We obtain new zero-momentum excited-states eigenfunctions with $2R' < 2R$ bosons and $N - 2R' > N - 2R$ fermions.

The case $Q = 0$ seems at this point slightly problematic: by taking the contours as in the previous sub-sections, we obtain new excited states above the $2R$ -boson, $N - 2R$ -fermion ground states, with energies

$$E_N^{(0,R)} + \frac{Ns^2}{2} \left[1 - \left(1 - \frac{2R}{N} \right)^2 \right] \quad (3.44)$$

that form a continuum. Indeed, here s does not seem to be restricted by any condition for having non-trivial contours, and since the wavefunction has zero momentum, there are no conditions coming from imposing that it be defined on the circle. We do not know how to interpret this continuum of zero-momentum solutions, if really it occurs; a more involved analysis of the explicit integral formulae would certainly be useful for this purpose.

3.4. Interpretation via Coulomb gas formalism of CFT

The goal of this sub-section is to clarify our construction in relation to the Coulomb gas formalism of CFT.

In the Coulomb gas formalism, one first constructs (boundary) vertex operators $V_p(\theta)$ (they are operators that act on the Hilbert space of radial quantization of CFT) with dimensions $p^2 - 2pq$ for some fixed q , and with ‘charge’ p . The charge of a vertex operator is just the associated eigenvalue of a charge operator Q (that is, $[Q, V_p] = pV_p$), that is supposed to exist on the Hilbert space. This means that the product $V_{p_1}V_{p_2}$ has charge $p_1 + p_2$, and taking into consideration the dimension, we have the OPE’s

$$V_{p_1}(\theta_1)V_{p_2}(\theta_2) \sim (\theta_1 - \theta_2)^{2p_1p_2} V_{p_1+p_2}(\theta_2).$$

The characteristic properties of these vertex operators are that correlation functions of products of such objects are nonzero only when the total charge is equal to $2q$, and that they evaluate, in our context, to the product of all pairings of the vertex operators involved, a pairing being just equal to $[2 \sin((\theta_1 - \theta_2)/2)]^p$ where p is the power of $\theta_1 - \theta_2$ that appears in the leading OPE.

One then constructs certain special level-2 null fields ϕ (and higher-level null fields as well) by choosing $q = \frac{\kappa-4}{4\sqrt{\kappa}}$ and identifying $\phi = V_p$ with $p = \frac{1}{\sqrt{\kappa}}$. This indeed reproduces the correct OPE’s of such null fields, but without the identity component; hence, these are very special level-2 null-fields.

In our case, we take these boundary fields and put at the centre of the disc the product of bulk holomorphic and anti-holomorphic vertex operators $V_P\bar{V}_P$, with charge $P = -Np/2 + q$ if there are N boundary null fields. By mapping the boundary theory to a holomorphic theory on the full plane (where \bar{V}_P becomes V_P at infinity), we are left with correlation functions of holomorphic vertex operators. The total charge requirement is satisfied, hence correlation functions are nonzero. This indeed reproduces the fermionic correlation function G_N^f (with an appropriate normalization), and in particular, one can check that the dimension of the product of bulk vertex operators is d_N^f .

The Coulomb gas construction then goes on to construct more complicated null-fields by inserting a vertex operator of dimension 1 and by integrating its position over closed contours. This insertion scales as a dimension 0 non-local object, and its effect is to change the fields that are involved. More precisely, the null fields become different null fields (with different OPE’s that contain the identity field), and the bulk field is modified. There are two possible dimension-1 vertex operators: $V_{r_{\pm}}$ with $r_{\pm} = q \pm \sqrt{q^2 + 1}$. The correlation function (3.1) is exactly the correlation function with one such insertion, and the function (3.25) is the correlation function with M insertions. Our derivation shows how these insertions modify the fields for various choices of the contours.

4. Completeness in the case $N = 3$ spinless

In this section, we consider in general the problem of determining if an eigenfunction of the Calogero–Sutherland Hamiltonian can give a boundary CFT correlation function G in the case of three particles. We show that mixed boundary conditions with $N = 3$ impose the dimension of the field to be d_1^f and, in the cases where $\kappa \neq 6$ and $\kappa \neq 8/n$ with $n = 2, 3, \dots$, we argue that the solution with any kind of mixed boundary condition has a 3-dimensional basis composed by the solutions with purely bosonic behaviours at some pair of colliding angles that we described in the previous section.

We will start with considerations for general particle number N . For definiteness, consider the ordering of the angles to be $\theta_1 > \theta_2 > \dots > \theta_N$ and consider the behaviour as $\theta_1 \rightarrow \theta_2$: it is a linear combination of the power laws $(\theta_1 - \theta_2)^{-2r_f}$ and $(\theta_1 - \theta_2)^{-2r_b}$.

As we said in the introduction, the constraints that come out of the system of equation (2.5) are essentially part of the general theory of null-vectors in CFT. In particular, the fermionic behaviour (1.3) corresponds to a fusion into a level-3 degenerate boundary field ($\phi_{1,3}$ in the Kac classification), and the bosonic behaviour (1.2) to a fusion into the identity (1) operator.

4.1. Conditions from null-vector equations

It will be convenient to consider the separation Δ between d_Φ and the 1-leg exponent d_1^f (2.20), that is, the equations $\mathcal{D}_j G = (d_1^f + \Delta)G$. We first look at arbitrary N . A generic solution to the Calogero–Sutherland system has, around $\theta_1 = \theta_2$, a basis of the form

$$G = \theta_{1,2}^{-2r} (A + B\theta_{1,2} + C\theta_{1,2}^2 + D\theta_{1,2}^3 + \dots) \tag{4.1}$$

where $r = r_f$ or $r = r_b$, where we use

$$\theta_{j,k} = \theta_j - \theta_k \tag{4.2}$$

and where $A \neq 0, B, \dots$ are functions of $\theta_2, \dots, \theta_N$. (That is, in general G can be a linear combination of one expansion with $r = r_f$ and one with $r = r_b$.) Note that for $\kappa = 8/n$ with $n = 1, 2, 3, \dots$, we have ‘resonances’: $-2r_f = -2r_b + n$; however, we will not look at the resulting logarithmic behaviours.

As shown in appendix C, the null-vector equations (2.5) lead to the following constraints, which can as well be seen as coming from null-vector CFT considerations:

$$(r = r_f \text{ and } \partial_2 A = 2B) \text{ or } (r = r_b \text{ and } \partial_2 A = 0 \text{ and } (B = 0 \text{ or } \kappa = 4)), \tag{4.3}$$

$$\sum_{k \neq 1,2} (f_{2k} \partial_k - h f'_{2k}) A - \frac{1}{6} (2r - h) A + 2\partial_2 B - \frac{(2r\kappa - \kappa - 6)(r\kappa - \kappa + 1)}{\kappa} C = \Delta A \tag{4.4}$$

and

$$r = r_b \Rightarrow -\frac{\kappa}{2} \partial_j^2 A + \sum_{k \neq 1,2,j} (f_{jk} \partial_k - h f'_{jk}) A = \Delta A \tag{4.5}$$

$$r = r_f \Rightarrow -\frac{\kappa}{2} \partial_j^2 A + \sum_{k \neq 1,2,j} (f_{jk} \partial_k - h f'_{jk}) A + f_{j2} \partial_2 A - 2h_{3,1} f'_{j2} A = \Delta A. \tag{4.6}$$

Equation (4.5) is the equation $\mathcal{D}_j^{(N-2)} A = (d_1^f + \Delta)A$ with the differential operator $\mathcal{D}_j^{(N-2)}$ being like \mathcal{D}_j but for the $N - 2$ angles $\theta_3, \dots, \theta_N$, instead of the N angles. Also, in (4.6),

$h_{1,3} = \frac{8-\kappa}{2\kappa}$ is the dimension of a level-3 degenerate field. Equations (4.5) and (4.6) indicate that the function A describes, in the case $r = r_b$, a correlation function with $N - 2$ level-2 degenerate boundary fields, and, in the case $r = r_f$, a correlation function with one level-3 degenerate boundary field (at θ_2) and $N - 2$ level-2 degenerate boundary fields.

According to (4.3), when the boundary fields fuse to the identity, the second term of the expansion, with coefficient B , is absent, except possibly when $\kappa = 4$. The case $\kappa = 4$ corresponds to the theory with $c = 1$, which is the free massless boson, where there is a natural operator of dimension 1 which indeed occurs as a symmetry descendant of the identity operator.

Along with conditions (4.3), equation (4.4) fixes C in terms of the function A (and the number Δ) in the fermionic and bosonic cases with $\kappa \neq 4$. For $\kappa = 4$, the function C also depends upon B , which is not necessarily zero.

4.2. Case $N = 3$

We now analyse in more detail the case $N = 3$. For spinless Φ , the function A depends only upon $\theta_2 - \theta_3$.

Let us first analyse the bosonic case. Then, equation (4.5) implies that $\Delta = 0$ since A is constant. Further, equation (C.2) fixes C uniquely (up to normalization):

$$C = \frac{h}{8 - 3\kappa} \left(f'_{23} + \frac{1}{6} \right) A.$$

It is simple to see that all coefficients D, \dots are then also fixed uniquely, if the solution exists, as long as $\kappa \neq 8/n$ for $n = 2, 3, 4, \dots$. It is worth noting that since a solution with one purely bosonic behaviour as some pair of colliding angles is unique, then a solution with two purely bosonic conditions must have all bosonic conditions by cyclic permutations.

When $\kappa = 4$, still in the bosonic case, a further analysis shows that in fact we must have $B = 0$, so that C and all other coefficients are also fixed uniquely, if the solution exists. Then, there cannot be non-logarithmic solutions with purely fermionic behaviour (and $\Delta = 0$), since the fermionic exponent occurs in the bosonic solution (a resonance). Hence, any non-logarithmic solution must be purely bosonic everywhere, but we showed that such solutions to the Calogero–Sutherland system do not satisfy (2.5) at $\kappa = 4$. That is, a bosonic solution will also have to involve logarithms.

When $\kappa = 8/3$, equation (C.2) becomes inconsistent, as it requires $A = 0$: a bosonic behaviour for this value of κ will have to involve some logarithms as well.

Similarly, when $\kappa = 8/n$ for $n = 4, 5, 6, \dots$, difficulties appear when trying to determine the coefficients D, \dots , and logarithms will be necessary.

Finally, it is worth noting that for $\kappa = 6$, since the solution is unique, it is given by the constant solution $G = \text{const}$.

Hence, we have showed that for $\kappa \neq 8/n$ for $n = 2, 3, \dots$, if a general solution has some contribution to a bosonic behaviour at any pair of colliding angles, say at $\theta_1 = \theta_2$, it must correspond to an operator of dimension d_1^f , and that the purely bosonic contribution at $\theta_1 = \theta_2$ is unique up to normalization. In the previous section, we constructed explicitly the unique solutions that are purely bosonic as some pair of colliding angles, and we saw that, except for $\kappa = 6$, they have fermionic contributions at other pairs of colliding angles. Now, the part of a general solution that is purely fermionic at $\theta_1 = \theta_2$ is not uniquely fixed. This part is fixed once the value of the function A (involved in its expansion) is fixed. But the function A is ruled by (4.6), which determines the possible behaviours at colliding angles $\theta_2 - \theta_3 = 0, 2\pi$ in accordance to the fusion rules $\phi_{1,3} \times \phi_{1,2} \mapsto \phi_{1,4}$ and $\phi_{1,3} \times \phi_{1,2} \mapsto \phi_{1,2}$. Moreover, one can

see that a choice of the ratios V between the amplitudes of these two behaviours as $\theta_2 \rightarrow \theta_3^+$, for instance, along with the eigenvalue Δ completely fix the solution up to a normalization. Since we have $\Delta = 0$, we are left with a one-dimensional subspace of solutions for A (up to normalization). We already know of such a one-dimensional subspace: it comes from taking linear combinations of the particular solutions G (constructed in the previous section) with purely bosonic behaviour at $\theta_2 = \theta_3$, those with purely bosonic behaviour at $\theta_3 = \theta_1$, and those with purely bosonic behaviour at $\theta_1 = \theta_2$, with the constraint that the behaviour at $\theta_1 = \theta_2$ of the resulting linear combination is purely fermionic. Hence, this constitutes all possible solutions with $\Delta = 0$ that are purely fermionic at $\theta_1 = \theta_2$. In other words, any general solution G that has some part of a bosonic behaviour at some colliding angles should be a linear combination of the three unique solutions that have purely bosonic behaviour at the three different pairs of colliding angles. This argument breaks down at the value $\kappa = 6$, since then only $G = \text{const}$ can have pure bosonic behaviour at some colliding angles.

5. Discussion

As we mentioned, our results are based solely on level-2 null vector equations of boundary CFT. Here, we attempt an interpretation of our results as measures in the continuum $O(n)$ loop model at criticality [12] (which we recall below), mainly based on the values of the exponents that we found.

In order to calculate prescribed measures from the system of differential equations, one needs to specify the boundary conditions: the various proportions of bosonic and fermionic behaviours at different pairs of colliding angles. Two problems arise.

The general problem of determining what boundary conditions completely fix the solutions is quite involved; in the case $N = 2$ it is the (solved!) problem of boundary conditions for second-order differential equations, and in the case $N = 3$ we solved it above (although not to mathematical rigor). In the general case, it is related to finding OPE's of null fields that form a consistent operator algebra. It is believed [5] that if one specifies all pairs, say P , where a bosonic behaviour occurs, along with some normalization condition, then the solution to all null-vector equations is fixed, and in particular the fermionic components at the pairs P is fixed. But in our case, the boundary conditions are specified quite differently (in particular, we specify both bosonic and fermionic components at many pairs).

More importantly, the problem of relating a set of boundary conditions (or an operator algebra) to prescriptions on measures is still far from being solved. Very natural arguments were given in [5] for the case where no bulk field is present, from SLE ideas. We will use these arguments below in a simplified version and in a different language (without using SLE ideas) along with some of our solutions in order to derive conjectures for certain measures in the $O(n)$ model.

5.1. Overview of the $O(n)$ loop model

A measure in the lattice $O(n)$ model has the form

$$\sum_{\mathcal{G}} x^\ell n^\omega \tag{5.1}$$

where \mathcal{G} denotes all configurations of self and mutually avoiding loops and, possibly, curves with prescribed end-points on the honeycomb lattice, ℓ is the total length of a configuration, ω is the total number of loops of a configuration, and $x = 1/\sqrt{2 + \sqrt{2 - n}}$ [12] is the value at which the system is critical (on the honeycomb lattice), for real numbers $-2 < n < 2$. We will

imagine restricting all loops and curves to lie inside a disc. The continuum limit is obtained by taking the lattice mesh size infinitely small (equivalently, by taking the disc infinitely large in units of lattice spacing). When the continuum limit of this model is taken, it is expected, if it exists, to be described in some way by a conformal field theory with central charge c given in (2.2) where κ is related to n via

$$n = 2 \cos \pi \left(1 - \frac{4}{\kappa} \right). \quad (5.2)$$

In the continuum limit, the resulting curves are expected to be described by SLE_κ curves. For the sake of keeping the discussion concise, in the following we will not think in terms of SLE curves (or in terms of growing such curves from some arbitrary points), but rather we will simply draw our intuition from the idea of the continuum limit of the $O(n)$ model.

Taking the continuum limit of a certain measure of the $O(n)$ model requires an appropriate re-normalization of this measure. For instance, the measure (5.1) for configurations with only one curve (apart from the loops) that starts and ends on fixed points on the boundary of the disc becomes infinite as the mesh size is made smaller. The series of numbers obtained as the mesh size is made smaller are quite meaningless. But if we take the ratio of that measure with another measure where the curve starts and ends on *different* fixed points, then we expect the limit of zero mesh size to be finite. This ratio is expected to be equal, in the limit, to a ratio of correlation functions in CFT, where level-2 null fields are inserted on the boundary of the disc at the positions where the end-points of the curves lie. We will normalize measures for single curves starting and ending at fixed points by always taking the ratio with, say, the measure where the fixed points are exactly opposite each other on the boundary of the disc. The result is what we will refer to as a measure on such configurations in the continuum, and is what corresponds to correlation functions of null fields up to a positive (nonzero) normalization. For more curves and other prescriptions on their shapes, we will keep the same principle: a measure will be a limit that only encodes the dependence on the starting and ending points of the curves, obtained by taking the ratio with such a measure where end-points are at arbitrarily chosen fixed positions. The limit is that of mesh size going to zero, and then of other parameters going to zero if necessary for the definition of the bulk field or of new boundary fields. The results of such normalized limits correspond to correlation functions.

It is worth noting that in the continuum $O(n)$ loop model, one can define fields $\mathcal{O}_{n'}(x)$ by the fact that, in the underlying lattice model, loops around the point x are counted with the value n' replacing the value n in the partition function. The dimensions of these fields was calculated in [14], and are given by

$$d_{n',n} = \frac{(\kappa' - \kappa)(\kappa\kappa' - 2\kappa - 2\kappa')}{\kappa(\kappa')^2} \quad (5.3)$$

where $n' = 2 \cos \pi \left(1 - \frac{4}{\kappa'} \right)$ (this will be used in the discussion in appendix D).

5.2. Fusion and measures

As two angles collide, two power law behaviours for the measure are possible (here we disregard possible logarithmic behaviours and resonances), and generically occur in linear combinations. The coefficient of the leading term of each power law is itself another measure, for different curve configurations. These new measures correspond to new correlation functions, where the two level-2 null fields have been replaced by a single field, as occurring in their fusion: $\phi \cdot \phi = \mathbf{1} + \phi_{1,3}$. The field $\mathbf{1}$ is the identity field and is associated with the bosonic behaviour, and $\phi_{1,3}$ is a level-3 null field and is associated with the fermionic behaviour. It

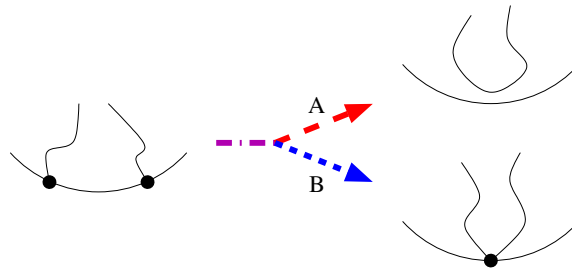


Figure 6. Interpretation of bosonic (A) and fermionic (B) behaviours in a generic boundary condition involving a linear combination of these behaviours.

is important to realize that only the fact that the field ϕ gives rise to the level-2 null vector equations (2.5) implies that $\mathbf{1}$ is the identity and that $\phi_{1,3}$ is a level-3 null field whose properties are essentially expressed in (C.4), (4.5) and (4.6). What curve configurations are associated with these correlation functions obtained by fusion?

We give here only heuristic arguments. First, the measure resulting from the fusion to the identity is that on configurations where the curves touching the boundary at the colliding angles are ‘disconnected’ from the boundary and joined, near to the boundary, into one curve (figure 6(A)). Of course, such an operation is quite imprecise, but we will only discuss qualitative features. Note that in order for this fusion to occur, it must be, in some sense, that the resulting curve does not affect the measure that corresponds to the correlation function obtained by removing the two colliding null fields, except for a possible normalization (since the operator resulting from the fusion is the identity).

Second, the measure resulting from the fusion to the level-3 null field is that on configurations with the additional prescription that two curves start at the fused point (figure 6(B)).

In appendix D, we use these general ideas to explain various known exponents for the case $N = 2$.

5.3. The fermionic ground states and the N -leg exponents

The correlation function G_N^f (2.19) can be interpreted using the fact that the associated bulk field dimension d_N^f (2.20) is the N -leg exponent [11]. That is, consider the measure on N curves that have end-points at the angles $\theta_1, \dots, \theta_N$ on the boundary of the disc and at some angles (such that curves do not cross each other) at a radius ε from the origin, as depicted in figure 7. As ε is sent to zero, this measure vanishes as $G_N^f \varepsilon^{d_N^f}$, up to a normalization.

The fact that this situation corresponds to uniform boundary conditions of fermionic type is easy to understand. Indeed, the bosonic behaviour is divergent when $\kappa < 6$, but there is no reason, as two points collide, for the measure to grow. In contrast, it should decrease since there are less and less configurations as the curves emanating from the colliding points are more and more constrained by each other. Hence there is no bosonic behaviour. Another way of understanding, valid for any κ , is as follows. As two angles collide, the curve resulting from a bosonic behaviour would be one that starts and end at the centre and that go all the way to a region not far from the boundary. But a curve that starts and ends at the centre has overwhelming probability of staying near the centre; imposing that it goes to a region near the boundary changes the measure, hence the fusion cannot give the identity operator; this is inconsistent, so there cannot be bosonic behaviour (see figure 8).

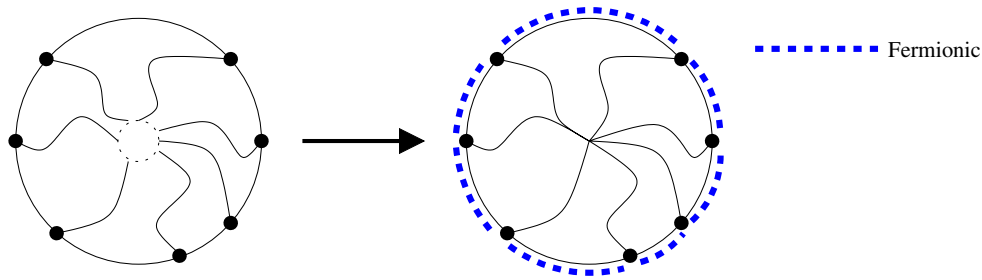


Figure 7. A typical configuration for the measure giving the N -leg exponent with $N = 7$. The inner circle has radius ε , and the leading part as it is sent to 0 is the correlation function G_7^f times the power law $\varepsilon^{d_7^f}$. The correlation function has all fermionic boundary conditions.

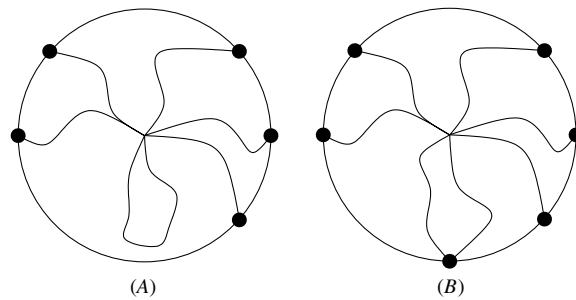


Figure 8. A typical configuration for the measure that would result as the coefficient of (A) a bosonic behaviour and (B) a fermionic behaviour in the measure giving the N -leg exponent with $N = 7$, as the two downmost end-points are approached in figure 7. The bosonic behaviour is forbidden, since the extra loop connected to the origin affects the measure, but the fermionic behaviour is allowed since the two curves connecting a common point to the origin extend macroscopically into the disc.

5.4. The solutions with mixed boundary conditions and the N' -leg exponents

The correlation functions $G_{C^{(i_1)}, \dots, C^{(i_R)}}$ (3.33) should be interpreted similarly using the fact that the associated bulk field dimension d_{N-2R}^f (2.20) is the $(N - 2R)$ -leg exponent. One could then expect that it is the measure on configurations like that of figure 2 (at least for $2R < N$; we will come back to the 0-leg exponent). Arguments like those of the previous sub-section indeed suggest, for $\kappa < 6$, that as two paired angles (two bosons, paired by a curve joining them in the continuum $O(n)$ model) collide, the measure should grow very much, since the curve can be made smaller and smaller. Hence, there should be a bosonic behaviour. However, it also suggests that the boundary condition as a paired angle (a boson) collides with an unpaired angle (a fermion) is the purely fermionic one: indeed, there is no reason for the measure to grow there, it should just decrease, as the curves are constrained by each other. This suggests that our solutions (3.33) are not the correct ones, and that we have to take linear combinations of these solutions to obtain the desired behaviours (if possible).

When there is just one pair of bosons and N particles, it is indeed possible to take linear combinations of our N -independent solutions (where N different pairs are taken) to obtain the suggested behaviour: bosonic at a single pair, say at the collision $\theta_1 \rightarrow \theta_2$, with some fermionic component, and purely fermionic everywhere else. For a bosonic behaviour at the

collision $\theta_1 \rightarrow \theta_2$, the linear combination $M_N^{(1)}$ is obtained from the inverse of an N by N matrix through

$$M_N^{(1)} \propto (1, 0, 0, \dots, 0) \begin{pmatrix} A & B & 0 & 0 & \cdots & 0 & B \\ B & A & B & 0 & \cdots & 0 & 0 \\ 0 & B & A & B & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B & 0 & 0 & 0 & \cdots & B & A \end{pmatrix}^{-1} \begin{pmatrix} G_{\mathcal{C}^{(1)}} \\ G_{\mathcal{C}^{(2)}} \\ \cdots \\ G_{\mathcal{C}^{(N)}} \end{pmatrix}$$

where

$$A = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma(1 - \frac{4}{\kappa})^2}{\Gamma(2 - \frac{8}{\kappa})}, \quad B = 2 \sin \frac{4\pi}{\kappa} \frac{\Gamma(1 - \frac{4}{\kappa}) \Gamma(-1 + \frac{8}{\kappa})}{\Gamma(\frac{4}{\kappa})}.$$

With appropriate normalization, it can be written as

$$M_N^{(1)} = \sum_{i=1}^N x_i G_{\mathcal{C}^{(i)}}, \quad x_i = \sum_{j=0}^{[N/2]} c_{i,j} A^j B^{[N/2]-j}$$

where the $c_{i,j}$'s are the unique solution to

$$c_{i,j+1} + c_{i+1,j} + c_{i+2,j+1} = 0, \quad (i = 1, 2, \dots, N - 1, j = -1, 0, \dots, [N/2])$$

with $c_{i+N,j} = c_{i,j}$ and

$$c_{1,[N/2]} = 1, \quad c_{2,[N/2]} = \cdots = c_{N,[N/2]} = 0, \quad c_{i,-1} = 0.$$

From formulae (3.34) and (3.35), one can check that this linear combination has purely fermionic behaviour as $\theta_i \rightarrow \theta_{i+1}$ for $i = 2, 3, \dots, N$ (with $\theta_{N+1} \equiv \theta_1 - 2\pi$), and a bosonic component as $\theta_1 \rightarrow \theta_2$ (along with some fermionic component). We have for instance

$$M_3^{(1)} = (A + B)G_{\mathcal{C}^{(1)}} - BG_{\mathcal{C}^{(2)}} - BG_{\mathcal{C}^{(3)}}.$$

As we said above, it is expected that this solution is, for every N , the unique one (up to normalization) with those behaviours—without the need to specify the fermionic component as $\theta_1 \rightarrow \theta_2$. The requirement for this solution to be a measure is that it be everywhere positive (with appropriate normalization). We see for instance that $M_3^{(1)} = (A - B)G_{\mathcal{C}^{(1)}}$ when all angles are equidistant, and that as $\theta_1 \rightarrow \theta_2$, we have $M_3^{(1)} \sim (A - B)(A + 2B)G_{N-2}^f (\sin \frac{\theta_1 - \theta_2}{2})^{-2r_b}$; at both of these particular points the function $M_3^{(1)}$ has the same sign. It is also possible to check numerically that everywhere it has the same sign. Hence it correctly represents a measure. We also verified that in the case $N = 4$ the sign is the same at the particular points where all angles are equidistant and where $\theta_1 \rightarrow \theta_2$. A general proof of positivity would be very interesting and would strengthen the conjecture according to which the linear combinations above are measures in the $O(n)$ model, but it is beyond the scope of this paper.

When more than one pair is taken, our solutions are not enough to form linear combinations with the suggested behaviours. One needs to take certain analytic continuations, which are not obviously real and positive.

Acknowledgments

We thank Denis Bernard for useful discussions and explanations. This work was supported by EPSRC under grants GR/R83712/01 (JC) and GR/S91086/01 (BD, post-doctoral fellowship).

Appendix A. A short definition of Schramm–Loewner evolution (SLE)

Radial SLE (which is the type of SLE of interest for our present work) is a way of constructing a measure $\mu(\gamma)$ for a random (non-self-crossing, continuous) curve γ on the unit disc \mathbb{D} joining a point $a \in \partial\mathbb{D}$ of its boundary to the centre of the disc, such that a certain property of ‘local conformal invariance’ holds. This property is mathematically known as domain Markov property, and says that

$$\mu|_{\Gamma \subset \gamma} = \mu \cdot f_\Gamma$$

where Γ is a curve with one end at the point a and the other inside the disc, and f_Γ is the uniformizing conformal map for Γ , a conformal map $f_\Gamma : \mathbb{D} \setminus \Gamma \rightarrow \mathbb{D}$ that maps the disc from which the ‘slit’ Γ has been removed back to the disc itself, preserving the centre (this map is defined up to a rotation). In the equation above, on the left-hand side one restricts the measure to curves γ that cover entirely Γ .

SLE is a construction of the measure μ through the stochastic growth of a curve from the point a to the centre. In general, the growth of a curve γ_t with ‘time’ $t \in \mathbb{R}^+$ can be described by the growth of its uniformizing conformal map g_t . The theory of Loewner says that with the uniformizing conformal map chosen to have real and positive derivative at the centre and with the parametrization of t given by $g'_t(0) = e^t$, it must satisfy the differential equation

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \frac{g_t(z) + a_t}{g_t(z) - a_t}$$

where $a_t \in \partial\mathbb{D}$ is a continuous function from \mathbb{R}^+ to the boundary of the disc. This driving function characterizes the growing curve that corresponds to the evolving conformal map g_t . When the curve is grown to $t \rightarrow \infty$, it connects the point $a = a_0$ to the centre of the disc. For the grown curve to be a random curve satisfying the property of conformal invariance above, Schramm [3] found that the random driving function must be a Brownian motion on the boundary of the disc:

$$a_t = e^{i\theta_t}, \quad \theta_t = \sqrt{\kappa} B_t + \theta_0$$

where B_t is a standard one-dimensional Brownian motion, with normalization $\mathbb{E}B_t^2 = t$. This describes a one-parameter family of measures, parametrized by $\kappa \in [0, 8]$, that satisfy the property of conformal invariance above; these are the only measures with this property.

The power of SLE comes from the fact that probabilities can be evaluated using the explicit growth process of the uniformizing map g_t . This generically gives rise to second-order linear differential equations which are of the form of level-2 null vector equations of CFT (see, for instance, the review [16]).

Appendix B. Derivation of the boundary level-2 null-vector equations on the disc

The covariance of the correlation function (2.1) under the transformation $z \mapsto z + \alpha(z)$ with (2.3) is found by inserting the appropriate charge:

$$\left\langle \phi(e^{i\theta_1}) \cdots \phi(e^{i\theta_N}) \left(\int_C T(z)\alpha(z) \frac{dz}{2\pi i} - \int_C \bar{T}(\bar{z})\overline{\alpha(z)} \frac{d\bar{z}}{2\pi i} \right) \Phi(0) \right\rangle$$

where C is a contour inside the disc $|z| < 1$ going round the origin counterclockwise once. Using the holomorphic OPE

$$T(z)\Phi(0) \sim \frac{h_\Phi}{z^2}\Phi(0) + \frac{1}{z}\partial\Phi(0) + \dots$$

and shrinking the contour C to the origin, we then get

$$(h_\Phi + \bar{h}_\Phi) \left(\sum_{j=1}^N b_j \right) G. \tag{B.1}$$

On the other hand, from the conformal boundary condition for a theory on the disc, the anti-holomorphic component $\bar{T}(\bar{z})$ of the stress–energy tensor inside the disc is related to the continuation of the holomorphic component outside the disc via

$$\bar{T}(\bar{z}) = \bar{z}^{-4} T(\bar{z}^{-1}). \tag{B.2}$$

Along with relation (2.4), deforming the contour towards the boundary of the disc gives

$$\int_C T(z)\alpha(z) \frac{dz}{2\pi i} - \int_C \bar{T}(\bar{z})\alpha(\bar{z}) \frac{d\bar{z}}{2\pi i} = \int_C T(z)\alpha(z) \frac{dz}{2\pi i} - \int_{C'} T(z)\alpha_j(z) \frac{dz}{2\pi i}$$

where the contour C' is outside the disc $|z| < 1$ and going counterclockwise. Hence, we are left with

$$- \oint_{z_1, \dots, z_N} \langle T(z)\phi(z_1) \cdots \phi(z_N)\Phi(0) \rangle \alpha(z) \frac{dz}{2\pi i} \tag{B.3}$$

where the integral means a sum of integrals counterclockwise around the points z_1, \dots, z_N .

Relation (B.2) specialized to the boundary $z = z_B$ with $|z_B| = 1$ implies a set of relations (a Virasoro algebra isomorphism preserving primary fields)

$$\bar{L}_n = (-1)^n \bar{z}_B^{2n} \sum_{k \geq 0} \bar{z}_B^k \frac{(2-n-k)_k}{k!} L_{n+k} \tag{B.4}$$

amongst the modes \bar{L}_n and L_n of the stress energy tensor,

$$T(z) = \sum_{n \in \mathbb{Z}} (z - z_B)^{-n-2} L_n, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} (\bar{z} - \bar{z}_B)^{-n-2} \bar{L}_n,$$

when they are applied on a boundary field ϕ at z_B . Along with the Ward identity associated with rotations,

$$(z_B L_{-1} - \bar{z}_B \bar{L}_{-1})\phi(z_B) = (z_B \partial \phi - \bar{z}_B \bar{\partial} \phi)(z_B)$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$, we then have for a primary boundary field

$$\begin{aligned} L_n \phi(z_B) &= 0 \quad (n \geq 1), & L_0 \phi(z_B) &= h \phi(z_B), \\ L_{-1} \phi(z_B) &= [z^h \partial (z^{-h} \phi(z))]_{z=z_B}. \end{aligned} \tag{B.5}$$

For a level-2 degenerate boundary field, on which $L_{-2} = \frac{\kappa}{4} L_{-1}^2$, this gives

$$L_{-2} \phi(z_B) = \frac{\kappa}{4} [z^h \partial^2 (z^{-h} \phi(z))]_{z=z_B}. \tag{B.6}$$

This can be used to evaluate (B.3), giving

$$\sum_{j=1}^N b_j z_j^h \tilde{\mathcal{D}}_j z_j^{-h} G$$

with

$$\begin{aligned} \tilde{\mathcal{D}}_j &= -\frac{\kappa}{2} \left(\frac{\partial}{\partial \theta_j} \right)^2 + \left(\frac{\kappa}{2} - 3 \right) i \frac{\partial}{\partial \theta_j} + \frac{6 - \kappa}{2\kappa} \\ &\quad - \sum_{k \neq j} \left(\cot \left(\frac{\theta_k - \theta_j}{2} \right) \frac{\partial}{\partial \theta_k} + i h \cot \left(\frac{\theta_k - \theta_j}{2} \right) - \frac{h}{2 \sin^2 \left(\frac{\theta_k - \theta_j}{2} \right)} \right). \end{aligned}$$

Using the similarity transformation

$$z_j^h \frac{\partial}{\partial \theta_j} z_j^{-h} = \frac{\partial}{\partial \theta_j} - ih,$$

we finally find the null-vector equations for the correlation function G to be (2.5) with differential operators (2.6).

Appendix C. Derivation of the constraints from null-vector equations

The equation $\mathcal{D}_1 G = (d_{\Phi_1} + \Delta)G$ leads to two equations, upon equating the coefficients of $\theta_{1,2}^{-2r-1}$ and $\theta_{1,2}^{-2r}$:

$$\partial_2 A - \frac{(r\kappa - 3)(2r\kappa - \kappa + 2)}{2\kappa} B = 0 \tag{C.1}$$

and (4.4). On the other hand, the equation $\mathcal{D}_2 G = (d_{\Phi_1} + \Delta)G$ leads to two similar-looking but different equations:

$$r\kappa \partial_2 A + \frac{(r\kappa - 3)(2r\kappa - \kappa + 2)}{2\kappa} B = 0 \tag{C.2}$$

and

$$\sum_{k \neq 1,2} ((f_{2k} \partial_k - hf'_{2k})A - \frac{1}{6}(2r - h)A - \frac{\kappa}{2} \partial_2^2 A - \kappa(2r - 1) \partial_2 B - \frac{(2r\kappa - \kappa - 6)(r\kappa - \kappa + 1)}{\kappa} C = \Delta A. \tag{C.3}$$

Equations (C.1) and (C.2) imply (4.3). On the other hand, it is a simple matter to check that these conditions automatically lead to the consistency of equations (4.4) and (C.3)

Thirdly the equations $\mathcal{D}_j G = (d_{\Phi_j} + \Delta)G$ for $j \geq 3$ lead to

$$-\frac{\kappa}{2} \partial_j^2 A + \sum_{k \neq 1,2,j} (f_{jk} \partial_k - hf'_{jk})A + f_{j2} \partial_2 A - 2(h - r) f'_{j2} A = \Delta A. \tag{C.4}$$

In the bosonic case $r = r_b = h$, this along with condition (4.3) simply gives (4.5), and in the fermionic case $r = r_f$, we find (4.6).

Appendix D. The case $N = 2$

From the viewpoint of the Calogero–Sutherland Hamiltonian, the case $N = 2$ is not of great interest. Indeed, since the eigenfunctions just depend on the single variable $\theta_1 - \theta_2$, it is a simple matter to obtain a general solution to the Calogero–Sutherland eigenvalue equation. Allowing arbitrary boundary conditions both as $\theta_1 \rightarrow \theta_2^+$ and as $\theta_1 \rightarrow (\theta_2 + 2\pi)^-$, any eigenvalue can be obtained (we do not discuss issues associated with the Hermiticity of the Hamiltonian in such conditions). Moreover, the two null-vector equations are equivalent, hence such a general solution satisfies all required properties of conformal correlation functions. Any bulk field dimension d_Φ is then allowed to appear. However, of course, not all are expected to correspond to dimensions of actual fields of the underlying CFT. It is then instructive to enumerate and interpret some scaling dimensions associated with known fields.

Besides the 2-leg exponent discussed above, there are three scaling dimensions known to correspond to well-defined $O(n)$ configurations that we wish to discuss.

One is the dimension 0, corresponding to the indicator event: it is associated with the measure on curves started at some angle θ_1 and ended at θ_2 that enclose the origin. Of course,

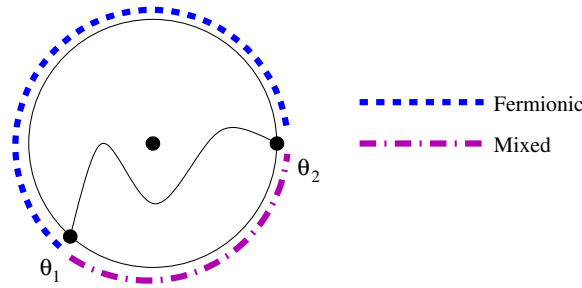


Figure D1. A typical configuration for the measure associated with the indicator event, with boundary conditions.

no ‘shrinking’ disc around the origin is involved in the definition of this measure, hence the associated exponent is trivially 0. The corresponding appropriately normalized correlation function gives Schramm’s formula [3] (on the disc), derived in the context of SLE:

$$G_2^{\text{Schramm}} = e^{i\frac{\theta}{2}} \sin^{\frac{2}{\kappa}}\left(\frac{\theta}{2}\right) {}_2F_1\left(1, \frac{4}{\kappa}; \frac{8}{\kappa}; 1 - e^{i\theta}\right), \quad \theta = \theta_1 - \theta_2.$$

By definition, at $\theta = 0$ the hypergeometric function is on its principal branch. It has a branch point at its argument equal to 1, hence as θ goes from 0 to 2π , the branch point is circled once counterclockwise and a monodromy is acquired.

The corresponding Calogero–Sutherland eigenfunction has purely fermionic boundary condition on the side where the SLE curve surrounds the origin ($\theta \rightarrow 0^+$), and a mix of bosonic and fermionic on the other side ($\theta \rightarrow 2\pi^-$), as depicted in figure D1. Again, the boundary conditions can be made plausible. On the purely fermionic side, the bosonic behaviour does not occur because it would mean imposing a macroscopic loop (since the loop has to be near to the boundary and has to surround the origin); such macroscopic loops do not occur with probability 1 in the measure on loops. On the other side, the bosonic behaviour does occur, because imposing a loop of any size not surrounding the origin does not affect the measure; they do occur with probability 1. The fermionic fusion also occurs on both sides since the two curves starting at one point are allowed to extend macroscopically in both cases.

Another is the dimension

$$d_2^m = \frac{(6 - \kappa)(\kappa - 2)}{8\kappa} + \frac{\kappa^2 - 16}{32\kappa} \tag{D.1}$$

(corresponding to the eigenvalue $E_2^m = \frac{1}{16}$ of the Calogero–Sutherland Hamiltonian). It is natural to consider this dimension, since the associated correlation function,

$$G_2^{1\text{-arm}} = \sin^{-2r_f} \left(\frac{\theta_1 - \theta_2}{4}\right) \cos^{-2r_b} \left(\frac{\theta_1 - \theta_2}{4}\right) \tag{D.2}$$

gives purely fermionic boundary condition on one side, and purely bosonic on the other side.

As was noted in [1], when specialized to $\kappa = 6$, it corresponds to the 1-arm exponent⁵, calculated in the context of SLE in [15]. More generally, for $\kappa > 4$ it gives $h/6 + c/12 + \lambda$ where λ occurs in the measure $\mu \propto \varepsilon^\lambda$ as $\varepsilon \rightarrow 0$ on single radial SLE curves that contain no counterclockwise loops around the origin before reaching a radius ε to the origin [15]. Hence, we expect that the quantity $G_2^{1\text{-arm}} \varepsilon^{d_2^m}$ give the leading part of the measure on single

⁵ The 1-arm exponent characterizes the power law with which vanishes the measure in site percolation with constraint that at least one path exists from the origin to a surrounding circle, as the radius of the circle is made infinite.

curves connecting points at angles θ_1 and θ_2 on the boundary of the disc with the condition that no loop forms around the origin unless it is completely contained inside a disc of radius ε around the origin. The extra terms $h/6 + c/12$ in d_2^m account for the change from a radial curve (starting on the boundary and ending at the centre) to chordal curve (starting and ending on the boundary).

This interpretation is corroborated by noticing that the dimension (5.3) of the fields \mathcal{O}_0 in the continuum $O(n)$ model is exactly the exponent d_2^m . Recall that the field \mathcal{O}_0 placed at the origin forbids any loop surrounding the origin in the $O(n)$ model, since it attributes to them a weight 0. Naturally, the dimension of this field at $\kappa = 6$ is the 1-arm exponent, since the absence of loops around the origin implies the presence of a percolation path from the boundary of the disc to the centre.

From this interpretation, the boundary conditions can be understood as follows. On the side of the bosonic behaviour, the fermionic fusion is absent because two curve starting from one point will almost surely, for $\kappa > 4$, have double points so that loops are formed around the origin; if this is forbidden, the two curve cannot extend macroscopically and the fusion does not occur. On the side of the fermionic behaviour, the bosonic behaviour is absent because joining the curve exactly produces a forbidden loop around the origin.

The last dimension that we wish to consider here is the 0-leg exponent ((2.20) with $N = 0$), which turns out to be $d_0^f = (c - 1)/12$. The corresponding correlation function can be read off from our solution with $N = 2$ and $R = 1$,

$$G_2^{0\text{-leg}} = e^{-i\frac{(\kappa+4)\theta}{2\kappa}} \sin^{1-\frac{6}{\kappa}}\left(\frac{\theta}{2}\right) {}_2F_1\left(1 - \frac{4}{\kappa}, 1 - \frac{4}{\kappa}; 2; 1 - e^{i\theta}\right) \quad (\text{D.3})$$

with $\theta = \theta_1 - \theta_2$ (this corresponds to the eigenvalue $E_2^{0\text{-leg}} = 0$ of the Calogero–Sutherland Hamiltonian). It gives purely bosonic conditions on one side ($\theta \rightarrow 0^+$), and mixed on the other side ($\theta \rightarrow 2\pi^-$). We expect that the quantity $G_2^{0\text{-leg}} \varepsilon^{d_0^f}$ is the leading part of the measure on configurations where a curve joins points at angles θ_1 and θ_2 on the boundary while being restricted not to come closer than ε to the origin. It is natural that this amplitude diverge as the radius is sent to zero (the 0-leg exponent is indeed negative, except when $\kappa = 4$, where it is 0). This interpretation is reinforced by noticing the following. It is a simple matter to observe that for the maximum value $n' = 2$ (that is, $\kappa' = 4$), one finds that the field $\mathcal{O}_{n'}$ of the continuum $O(n)$ model has a dimension (5.3) given by the 0-leg exponent $d_{2,n} = d_0^f$. For this maximum value, there is more likely a loop around the origin, which constrains the curve to stay away from the origin as described above. The boundary conditions can also be understood from this picture. On the bosonic side, there is no fermionic fusion because the two curves starting from one point are restrained away from the origin (they cannot form small enough loops around the origin). On the mixed side, fermionic contributions are clearly non-zero, and the bosonic fusion occurs since adding a macroscopic loop around the origin does not change the measure (such loops are already very likely).

Note that it is this situation that we generalized to N particles with a N' -leg bulk field, $N' = N - 2M$, $M \in \mathbb{N}$.

References

- [1] Cardy J 2004 *Phys. Lett. B* **582** 121
- [2] Sutherland B 2004 *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (Singapore: World Scientific)
- [3] Schramm O 2000 *Isr. J. Math.* **118** 221 (Preprint [math.PR/9904022](https://arxiv.org/abs/math.PR/9904022))
- [4] Cardy J 2003 *J. Phys. A: Math. Gen.* **36** L379
Cardy J 2003 *J. Phys. A: Math. Gen.* **36** 12343 (erratum)

- [5] Bauer M, Bernard D and Kytölä K 2005 Multiple Schramm–Loewner evolution and statistical mechanics martingales *Preprint* [math-ph/0503024](#)
- [6] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys. B* **241** 333
Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *J. Stat. Phys.* **34** 763
- [7] Dubédat J 2005 Euler integrals for commuting SLEs *Preprint* [math.PR/0507276](#)
- [8] Dotsenko S and Fateev V 1984 *Nucl. Phys. B* **240** 312
- [9] Dotsenko S and Fateev V 1985 *Nucl. Phys. B* **251** 691
- [10] Kytölä K 2006 Virasoro module structure of local Martingales for multiple SLEs *Preprint* [math-ph/0604047](#)
- [11] Saleur H and Duplantier B 2987 *Phys. Rev. Lett.* **58** 2325
- [12] Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062
- [13] Di Francesco P, Mathieu P and Senechal D 1997 *Conformal Field Theory* (Berlin: Springer)
- [14] Cardy J 2000 *Phys. Rev. Lett.* **84** 3507
- [15] Lawler G F, Schramm O and Werner W 2002 *Electron. J. Probab.* **7** 1–13 *Preprint* [www.math.washington.edu/~ejpecp/EjpVol7/paper2.abs.html](#)
- [16] Cardy J 2005 *Ann. Phys.* **318** 81